THEORY OF FRICTION.

A TREATISE

ON

THE THEORY OF FRICTION.

BY

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PREFACE.

The Theory of Friction, considered as a part of Rational Mechanics, has hardly received the attention which it deserves. Even in the most complete systematic treatises, the space accorded to its discussion is small, compared with that bestowed upon questions in which the hypothesis of perfect smoothness has been assumed as the basis of the investigation. And it seems probable that many students have been led to regard the discussion of this force, less as a part of Rational Mechanics, than as a correction to be applied before the investigations of that science can be made practically useful. Such an idea, if it exist, is a complete mistake. The theory of friction is as truly a part of Rational Mechanics as the theory of gravitation. The force with which this theory is concerned is subject to laws as definite, and as fully susceptible of mathematical expression, as the force of gravity. And even if the imperfection of our analysis should render impossible the actual solution of the problems which are presented to us in this theory, such an imperfection, which is purely mathematical, cannot affect the right of the theory itself to be considered as a part of Rational Mechanics. Neither is this right affected, even if it be shown that the laws usually assigned to the force of friction are not mathematically true for any known substance. Rational Mechanics,

considered as a hypothetical science, takes no account of this fact, and for the reality and usefulness of the conclusions it is only necessary that these laws should be approximately true, as they certainly are. Throughout the present treatise the proportionality of the force of friction to the pressure is assumed. This law, though not mathematically correct, represents the facts with sufficient exactness to serve as the basis of a theory whose results approximate closely to the truth. The adoption of a more complicated law would have greatly enhanced the mathematical difficulties of the theory, yet without giving results mathematically coincident with facts. Here, as in all other applications of Rational Mechanics to actual phenomena, we must be content with approximations to the truth.

The plan of the present work may be shortly stated as follows :—

Commencing with the two great classes of force, namely, moving and resisting forces, the Author has, in Chap. I., pointed out the important distinction between them, consisting in the fact that while forces of the first class are independent of the forces by which they are opposed, those of the second class vary in magnitude or direction, or both, with these opposing forces.

Two kinds of resisting force are then especially noticed, namely, 1. Forces by which the geometrical conditions, connecting with each other the several parts of a material system, may be replaced, without making any change in its motion (or rest, if it be at rest). These forces are distinguished by the name "geometrical forces," and their true character is pointed out. 2. Friction. The laws which govern this force are then given, and the action of a rough surface upon a particle

which rests or moves upon it is expressed by means of a geometrical conception which is (so far as the Author is aware) due to Mr. Moseley. The important differences between the friction of rest and the friction of motion are now noticed for the first time.

In Chap. II. the Author has considered the problem of equilibrium, noticing especially the *indeterminateness* which often appears in questions where friction is one of the acting forces. This apparent indeterminateness is traced to its source, namely, the abstractions of Rational Mechanics; and it is shown that if the conditions of the problem be stated as they really exist in nature, no such indeterminateness will appear. Various examples of the problem of equilibrium, both of particles and of solid bodies, are given.

Chap. III. treats of *extreme* positions of equilibrium, those, namely, in which the smallest diminution in the friction acting at some one or more points of the system will destroy its equilibrium. In such cases it becomes an important question: What points of the system are in extreme positions? or, in common words, what points of the system are so situated, that any diminution of the *roughness* of the surfaces which are there in contact would cause the system to *slip?*

Proceeding to investigate this question, the Author has given some examples, in which it admits a simple solution. The general problem is then examined by the help of certain analytical Lemmas, and rules are given for its solution. But the great length of the process would make this solution very difficult, except for the simplest cases.

A special section is devoted to extreme positions of a solid body.

Chap. IV. treats of the motion of a particle, or system of particles. Here the indeterminateness before noticed in statical problems appears again, and is particularly considered.

A special section is, on account of the importance of the question, devoted to the problem of *initial* motion, and another to the motion of a particle upon a surface which is itself moving.

Chap. V. treats of the motion of a solid body, with special reference to the case in which the motion is a pure rotation round an axis fixed or variable, and some general propositions are given applicable to this case. The case of a body which rolls without sliding, and the initial motion of a solid body, are also specially considered.

Chap. VI. treats of the distinction, peculiar to the present subject, between *necessary* and *possible* equilibrium. It is shown, by a consideration of the difference between statical and dynamical friction, that there are certain positions of a system in which equilibrium *must* exist, and others in which it *may* exist, but does not *necessarily* exist; and a method is given by which these two kinds of equilibrium may be distinguished from each other. This, although a question of equilibrium, and therefore appearing to belong properly to Statics, could not have been considered at an earlier part of the inquiry, inasmuch as, like the problem of stable and unstable equilibrium, it is solved by principles essentially dynamical.

In Chap. VII. the Author has considered the principles on which we may attempt to remove the ambiguity so constantly occurring in problems in which friction is one of the acting forces. He has shown that this can

be done only by rejecting the abstractions of Rational Mechanics, and considering the conditions of the problem as they really exist in nature. The mathematical difficulties which obstruct such an investigation are very great, but the Author has been able to give one example in which the solution can be effected, and which will sufficiently illustrate the principle of the method.

Chap. VIII. contains a number of miscellaneous problems, three of which have been discussed at some length, namely, the problem of the top, the problem of friction wheels, and the problem of locomotives. Subjoined is a collection of questions for exercise, suited to the younger students. Walton's admirable volume rendered it unnecessary to multiply these to any great extent.

Some of the principles contained in the present treatise are believed to be here enunciated for the first time, and the subject is, even in theory, very far from being exhausted. But if the Author shall have succeeded in so far attracting the attention of mathematicians to the properties of this remarkable force, as to secure for the theory a more ample discussion than has as yet been given to it, he will think that his labour has been well bestowed.

The Author is deeply indebted to the Rev. Richard Townsend, University Professor of Natural Philosophy, not only for his kindness in correcting the press, but also for valuable suggestions made during the progress of the work. He also desires to return his best thanks to the Board of Trinity College, for their liberality in aiding to defray the expenses of printing.

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THEORY OF FRICTION.

CHAPTER I.

DEFINITIONS AND PRINCIPLES.

1. THE forces which we find in nature appear to belong to two different classes. The former of these, comprising, among others, gravity, the elasticity of gas, and muscular force, are capable of producing sensible motion, and may therefore be termed *moving* forces. The second class, comprising the resistance of fixed obstacles, the resistance which fluid media oppose to the passage of bodies through them, and friction, are not capable of producing sensible motion, and show their effects only by the apparent destruction of the motion which has been caused by forces of the former class. These may be termed *resisting* forces.

Modern research has indeed made known to us that the destruction is only apparent. That which we term the destruction of motion by friction or resistance is in reality but the conversion of the sensible motion of the moving body into the vibration of the resisting obstacle, or into that other species of motion which we denominate heat. Still, however, the distinction between these two species of force is real. Resisting forces are incapable of *producing* sensible motion, and if they do not actually destroy it, they do convert it into a species of motion, which is, to sight, insensible.

It must be remembered, however, that the application of the term "moving forces" to those of the first class does not imply that these forces always act in producing rather than in destroy-

ing motion. Thus, when a ball is thrown vertically upwards, the velocity with which it leaves the hand is gradually destroyed by the combined forces of gravity, and the resistance of the air. Both forces are then in this case resisting or destroying forces. But there is an essential difference between the principles upon which these forces act in destroying motion. Gravity destroys the upward motion of the ball, because it has a tendency to generate motion in the opposite direction—a tendency which will, after a certain time, produce a visible effect in causing the ball to descend. But the resistance of the air has no such tendency. It is simply a destroying force, diminishing the *upward* motion of the ball while ascending, and diminishing its *downward* motion after it has begun to descend.

2. There is another important distinction between these two classes of force. Forces of the first class are independent, both in direction and in magnitude, of the forces by which they may be opposed. Thus, for example, the weight of a body is a perfectly definite force, not in any wise dependent upon any other force which may be used to overcome it. If the body be at rest, the force which its weight equilibrates must have a certain definite direction and magnitude. If it be in motion, the quantity of movement which is produced or destroyed by gravity is wholly independent of the other forces which may act upon it.

But it is otherwise with the forces which we have denominated "resisting forces." These are not independent of the opposing forces. The forces which they are severally capable of equilibrating are not definite, but are capable of being varied, sometimes within very wide limits, without disturbing the equilibrium of the system. In the case of the resistance of a medium, the quantity of movement destroyed by this force does depend upon the force by which the opposite movement is produced. In the case of a fixed obstacle, the resistance is always equal and opposite to the force by which it is sought to overcome it. And, as we shall presently see, the force of friction depends not only upon the nature of the surfaces which are in contact, but also upon the other forces which act upon the system.

The distinction here indicated may be expressed by saying

that forces of the first class are *in their nature determinate*, every such force being, for any given position of the system, capable of assuming but one value. Forces of the second class, on the other hand, are *in their nature indeterminate*, every such force being for a given position of the system capable of assuming an infinite number of values. We shall see further on that these forces are in certain cases capable of being determined by the equations of equilibrium or motion, and in other cases wholly indeterminate, unless their values be specially given.

3. The resistance of a fixed obstacle belongs to a class of forces which we may call, using the abstractions of Rational Mechanics, "geometrical forces." When a material particle belongs to a *system*, that is to say, when it is connected by certain geometrical relations with a number of other particles, we know that neither its motion, if it move, nor the conditions of its equilibrium, if it be at rest, are the same as they would be if the particle were independent of all others. The effect of the forces which act upon it is modified by the connexions which exist between it and the other particles of the system to which it belongs. Now, this modification may be always conceived to be produced by a force. So far, therefore, as questions of either equilibrium or motion are concerned, the geometrical relations may, without in any way altering the mechanical phenomena, be replaced by a system of forces, provided that these forces be suitably determined. These are what we have termed the "geometrical forces," which may be defined to be, *forces which are mechanically equivalent to the geometrical relations of the system.* The methods of ordinary mechanics furnish, as is well known, a fixed rule for determining the *line of direction* of each of these forces, these lines of direction depending only upon the nature of the geometrical relations which the forces replace. But the *magnitudes* of these forces are not determinable in the same way, depending as they do not only on the nature of the geometrical relations, but also upon the forces which act upon the different parts of the system.

Thus, for example, let it be supposed that the system consists of a single particle, and that the geometrical condition to which it is subjected is, that it shall always be found in a given

plane. Then we know that the force which is equivalent to this geometrical condition must be in *line of direction* perpendicular to the given plane, and in *magnitude* equal to the force which acts upon the particle resolved along that perpendicular. The actual direction of the geometrical or replacing force is immediately opposed to this component.

The principle contained in this example is general. The *line* of direction of the force which is equivalent to a geometrical condition depends solely upon that condition; the choice between the two directions contained in that line, as well as the actual magnitude of the force, depends also upon the other forces which act upon the particle.

4. Strictly speaking, the forces which we have termed "geometrical forces" have no real existence in nature. Their existence depends upon the abstractions of Rational Mechanics, which are never mathematically true. No system of really existing bodies will perfectly fulfil geometrical conditions. Thus, when we speak of a material particle as being in all its positions situated in a certain plane, it would be necessary to the perfect fulfilment of this condition that the supporting surface should be geometrically plane, and absolutely unyielding, neither of which conditions is fulfilled by any substance which we find in nature. The forces which are really brought into play by the connexions of a system depend upon the elasticity of the bars and strings and supporting surfaces, by which these connexions are established, and this elasticity is developed, not by the *fulfilment*, but by the *violation* of certain geometrical conditions.

Thus, for example, if the system be composed of two particles, connected by the geometrical condition that the distance between them shall be invariable, the methods of Rational Mechanics require us to replace, at each particle, this condition by a force whose line of direction passes through the two particles. But if we attempt to realize the condition by a connecting bar, the force which we thus introduce into the system, namely, the force of elasticity, is not brought into play until this condition is violated by the lengthening or shortening of the bar.

The validity of the conclusions of Rational Mechanics is not, however, in general impaired by this distinction, and it is pos-

sible to give a definition of the force developed by any of the connexions of a system, which is applicable either to the problem considered in the abstract, or to the problem as it really exists in nature. In either case, the force in question is one which *resists* the violation of the geometrical condition. Whether, therefore, the condition be perfectly fulfilled or not, this force will enter similarly into the statical or dynamical equations. Thus, for example, whether we regard the tension of a string as a force resulting from its inextensibility, or as an elastic force developed by an actual extension of the string, it is in either case a force which *resists* extension, and will enter similarly into the equations of equilibrium or motion of any system of which this string forms a part.

It is not of course true that the complete solution of such a problem is the same, whether the abstractions of Rational Mechanics be or be not assumed to be mathematically exact. The geometrical forces and the forces of elasticity do indeed enter similarly into the equations of equilibrium or motion, which have therefore the same form in either case. Hence it is plain that the equations which may be obtained by eliminating the geometrical forces in the one case, and the forces of elasticity in the other, are absolutely identical for both cases. But the remaining equations are different. These are, in the abstract problem, the geometrical equations themselves, which are here supposed to be true. In the physical problem, where these equations are not rigorously true, the remaining equations of the problem are derived from the supposition that the forces of elasticity, which here take the place of the geometrical forces, are given functions of the coordinates.

But although the solutions of the rational and physical problems respectively are not mathematically identical, it is easy to see that the solution of the rational problem is very approximately true for the physical problem. For the equations by which the rational problem is solved may, as we have seen, be divided into three groups. 1. Equations obtained by eliminating the geometrical forces between the equations of equilibrium or motion. 2. The geometrical equations themselves. 3. The equations which give the actual values of the geometrical forces.

Equations of the first group are strictly true in the physical as well as in the rational problem. Equations of the second group are, in all physical problems to which it is customary to apply the methods of Rational Mechanics, very approximately true. Hence the positions of the particles and the relations between the acting forces necessary for equilibrium, depending altogether upon equations of the first two groups, are very approximately the same for the physical and the rational problems; and the same is true for the movements of the particles, if the problem be dynamical. The remaining group of equations which in the rational problem determine the magnitudes of the geometrical forces, will, in the physical problem, determine the small extensions or contractions of the bars, strings, &c., which are necessary for the development of the equivalent elastic forces.

The definition which the principles of Rational Mechanics give of the geometrical forces of a system, although depending upon an unreal abstraction, is thus in general very approximately coincident in its results with the definition which would be derived from actual experiment. We shall show further on that this principle is liable to certain exceptions.

5. The distinction here indicated between the external forces which act upon a system and those which, following the abstractions of Rational Mechanics, we have called the "geometrical" forces, is, as we shall hereafter find, of the greatest importance. For the present, it will be enough to observe, that forces of the first kind are properly reckoned among the *data* of a mechanical problem. That is to say, the intensities and directions of these forces are given functions of the co-ordinates of the several particles. But it is otherwise with the geometrical forces. These are not, like the former, given functions of the co-ordinates. Their lines of direction are indeed deducible from these quantities by a known rule; but their intensities are themselves, unless their values be specially given, among the unknown quantities of the problem, to be determined by the equations of equilibrium or motion, if this be possible. We shall hereafter find cases in which, although the positions of the particles and the external forces acting upon them are fully known, the geometrical forces remain still indeterminate.

6. The other species of resisting force, which we propose to consider now, is Friction, or the resistance which we experience when we attempt to move one body along the surface of another against which it is pressed. Of this force there are two kinds, the difference between which depends upon the nature of the relative motion which we produce, or endeavour to produce.

(1.) If we attempt to make the one body *slip* upon the other; in other words, if we attempt to give to the point of the one body which is in contact with the second a movement *along* the surface of the second, we find that the attempt is resisted by a force which is called "friction of the first kind."

(2.) If we attempt to make the one body *roll* upon the other, that is to say, if the movement which we give, or attempt to give it, be a movement of rotation round an axis passing through the point of contact, and situated in the common tangent plane, this movement is resisted by a force which is called "friction of the second kind." We shall for the present confine our attention to the first of these two species.

7. The laws of friction of the first kind, as ascertained by experiment, are as follows:—

(1.) The line of direction of this force is always situated in the common tangent plane to the two bodies which are in contact with each other.

(2.) If the particle of the body or system which is in contact with the other be in motion along its surface, the force of friction is always directly opposed to this motion.

(3.) If the particle of the body or system, which is in contact with the other, be at rest, the force of friction is directly opposed to the tangential component of the resultant of all the other forces, external and geometrical, which act upon the particle.

(4.) If the particle be in motion, as in (2), and if P be the normal pressure existing between the two bodies, and F the force of friction, then

$$F = \mu P,$$

where μ is a numerical coefficient depending solely upon the nature of the surfaces in contact.

(5.) If the particle be at rest, as in (3), the force of friction may have any value from 0 to μP. The actual value which it has in any given case, and which may be termed *the effective force of friction*, is necessarily equal to the tangential component spoken of above, which component must therefore not exceed μP.

8. The action which a rough surface exercises on any particle or body which is in contact with it may be regarded as being compounded of two forces at right angles to one another, namely, (1). The reaction of the surface, as that term is commonly used. This force is directed in the normal to the surface, and is equal and opposite to the force which we have above denominated P. (2). The force of friction, F. This force must be directed in a line touching the surface, and may have any value from 0 to μP.

Let it be supposed that these two forces are compounded into one, which will then represent in magnitude and direction the complete action of the rough surface. Then it is evident that the magnitude of this force will be

$$\sqrt{P^2 + F^2},$$

and that its direction will make with the normal to the surface an angle which may have any value from 0 to $\tan^{-1}\mu$. This law may be geometrically expressed as follows:—

Conceive a cone of revolution to be described having its vertex at the given point, and of which the normal to the surface is the axis. Let the semi-angle of this cone be $\tan^{-1}\mu$. Then it is plain from the laws above stated that the complete resistance of a rough surface is limited by one only condition, namely, that its line of direction cannot lie outside the cone so described. This cone may be termed *the cone of resistance*. The semi-angle of the cone, being the angle whose tangent is equal to the numerical coefficient μ, is usually termed *the angle of friction*. We shall in general denote it by the symbol ε.

9. The numerical coefficient μ, which is called *the coefficient of friction*, depends, as has been said, upon the nature of the surfaces which are in contact. Experience has shown also that

it has different values for the cases of rest and motion. The force with which a rough surface resists the *commencement* of motion is, *cæteris paribus*, greater than that with which it resists the *continuance* of motion once commenced. Hence, inasmuch as the force is in both cases represented by μP, it is evident that μ must be greater when the particle is at rest than when it is in motion. This is expressed by saying that *the coefficient of statical friction is greater than the coefficient of dynamical friction.*

10. There is another and a more important difference between statical and dynamical friction. When a material particle moves upon a rough surface, the force of friction is, as we have seen, directly opposed to the motion of the particle. If, therefore, this motion be governed by any geometrical condition, which determines its line of direction on the supporting surface, the line of direction of the force of dynamical friction is also determined. Thus, for example, if the supporting surface be a plane, and if the particle be attached by a rigid, inextensible rod to a fixed point, the motion of the particle will necessarily be in a determinate circle. It is plain, therefore, that the line of direction of the frictional force which acts upon the particle, *while in motion*, is a tangent to this circle.

More generally, if a system of particles, whose positions are connected by certain geometrical relations, and each of which rests upon a rough surface, be in motion, we are not at liberty to assume, for the direction of the force of friction at each point, *any* line in the tangent plane at that point. For it is plain, from what has been said, that a system of possible lines of *direction* for the forces of friction must be coincident with a system of possible lines of *motion* of the several particles. Whatever limitations, therefore, are imposed upon these latter directions by the geometrical conditions of the system, the same limitations are necessarily imposed upon the directions of the forces of dynamical friction. But the same limitation does not hold with regard to the forces of *statical* friction, whose directions are not necessarily coincident with *any* system of displacements governed by the geometrical conditions.

For the present, it will be sufficient to notice this important distinction, which we shall discuss more fully in a subsequent chapter.

CHAPTER II.

EQUILIBRIUM WITH FRICTION.

I.—*General Principles of Equilibrium.*

1. WHEN the forces which act upon the particles of which a system is composed are definite functions of the co-ordinates of these particles, it is known that such a system will have in general a finite number of positions of equilibrium, separated from each other by finite intervals. For, let n denote the number of the particles of which the system is composed, and m the number of equations of condition: then we know that the number of equations of equilibrium will be $3n$, which, with the m equations of condition will give $3n + m$ equations in all. But this is also the number of the unknown quantities, namely, $3n$ co-ordinates of the particles, and m geometrical forces, of which the intensities only are unknown. Hence the proposition is evident.

If, however, any one or more of the forces be not determinate functions of the co-ordinates, the number of the unknown quantities will exceed the number of equations, and there will be in general an infinite number of positions satisfying the conditions of equilibrium, disposed in one or more groups, in each of which these positions succeed one another *continuously*. If, for example, any of the forces which we have termed *geometrical*, that is to say, the forces which are mechanically equivalent to the geometrical relations of the system, were indeterminate in direction as well as in magnitude, it is plain that the number of the unknown quantities would exceed the number of the equations, and therefore that the problem would be indeterminate. It is true, indeed, that no force which is, in this sense of the word, a geometrical force, is ever indeterminate in its direction, which can always be deduced by a well-known rule from the equation which it replaces. But the resistance of a rough sur-

face, which is intimately connected with one of the geometrical conditions of the system, *is* indeterminate both in magnitude and direction. The former of these is wholly indeterminate, and the second is restricted only by a certain limitation, namely, that it cannot lie outside a certain cone. Wthin this cone, the direction of the resistance is, like its magnitude, wholly indeterminate. We may therefore expect, as we shall, in fact, find it to be true, that when the system includes a rough surface, the positions of equilibrium will not be absolutely determinate, but will succeed each other continuously in one or more groups.

2. Thus, for exanple, if a material particle, acted on solely by gravity, be placed upon a fixed smooth surface, we know that it will only remain at rest if the position chosen for it be such that the tangent plane at this point is parallel to the horizon; for only at such a point can the force of gravity, which is necessarily in the vertical, be neutralized by the reaction of the surface, which is necessarily in the normal. But it is otherwise if the surface be rough. Here the reaction of the surface is not necessarily in the normal, but may make with that line any angle not exceeding a certain fixed limit. If then, round each of the points which are characterized by a horizontal tangent plane, we describe a curve along which the tangent plane makes with the horizon an angle equal to the limiting angle, each of these curves will in general include a space throughout which the tangent plane is inclined to the horizon at an angle less than the given limiting value. Throughout the whole of this space it is possible, without violating any of the conditions by which the force of friction is limited, to assign to that force such a magnitude and direction as shall neutralize the force of gravity, and thus allow the particle to remain at rest. Throughout the whole of this space, therefore, equilibrium is possible.

3. We have seen that the force of friction is *in its own nature* indeterminate, both in direction and in magnitude. But as absolute indeterminateness cannot really exist,* we must seek for some principle which will enable us to discover the actual direction

* It is necessary here to distinguish carefully between the different senses in which the word "determinate" may be used. 1. A force may be dependent solely

and magnitude which belong to this force in any given problem. This principle, which we obtain from experience, and which has been already enunciated in other terms, is as follows:—

The force of friction which acts at any point of a system will, if possible, assume such a magnitude and direction as will keep that point at rest.

The same principle is true for the normal resistance of the surface, and therefore for the force compounded of these two, which we have denominated "the resistance of a rough surface." We have, therefore, for the determination of this force, the following rule, which may also be called a principle of equilibrium.

Let all the forces, external and geometrical (excluding the action of the supporting surface), which act upon the given point be compounded into one, R. Then if the line of direction of this force lie within or upon the cone of friction, the point in question is in equilibrium, and therefore the resistance of the rough surface is equal and opposite to R.

Thus, for example, in the case of the gravitating particle which we have just considered, the region of *possible* equilibrium is shown by this principle to be also a region of *actual* equilibrium, the resistance of the rough surface being, throughout this region, equal and opposite to the force of gravity. If it be required to determine the force of friction separately, this is readily done by resolving the complete resistance of the surface perpendicular and parallel to the tangent plane. The latter component is the force of friction.

More generally, whatever be the force acting upon the particle, it will be in equilibrium, if the force make with the normal an angle not exceeding the angle of friction. Comparing this case with that of a particle situated on a smooth surface, we see that, whereas in the latter case *two equations* are necessary

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upon the *positions* of the several particles of the system, so that for any given position of the system it can have but one determinate direction and one determinate intensity. Such a force is, for example, the attraction of a centre. 2. A force may require for its determination not only a knowledge of the *positions* of the several parts of the system, but also of the other *forces* which act upon them. Forces belonging to the second class may be called indeterminate in their own nature; but no force is *absolutely* indeterminate. Such a supposition would plainly be inconsistent with the universality of law.—*Vid. inf.* § 9.

for the equilibrium of the particle, there is required in the former case only a single *condition*, not an equation. The physical reason of this difference is, that whereas the resistance of a smooth surface must necessarily act in one determinate line, and can therefore only equilibrate an external force acting in the same line, the resistance of a rough surface may, within certain limits, assume *any* direction, and will, within those limits, *necessarily* assume that direction which is requisite for equilibrium. The only condition of equilibrium is, therefore, that the applied force should not be without the prescribed limits.

4. If all the forces acting upon the particle (excluding the resistance of the rough surface) were of the kind which we have called external forces, their values would be necessarily determinate functions of the position of the particle. In this case, the resultant of these forces being also determinate, the principle stated above would enable us to assign the direction and magnitude of the resistance of the surface. But if any of these forces be geometrical, only their *directions* are given by the geometrical conditions which they replace. There is nothing, so long as we adhere to the abstractions of Rational Mechanics, to determine their *intensities*, which are therefore themselves among the unknown quantities of the problem. In such a case, therefore, we may naturally expect, as we shall in fact find it to be the case, that the solution furnished by these abstractions will not always be complete, some one or more of the unknown quantities remaining indeterminate.

Thus, for example, let the system be composed of two material particles, P, Q (Fig. 1), acted on solely by gravity, connected by a rigid weightless rod, and each of which rests upon a rough plane. Conceive a vertical plane to be drawn through the connecting rod. Let the plane of the paper represent this plane, and let AP, AQ be the sections of the supporting planes. Let G be the centre of gravity of the two particles, and GV a vertical. Let the cones of resistance be constructed at P, Q, and let Pp, Pq, Qp, Qq, be the sections of these cones made by the plane of the paper, which we shall suppose to cut them both. Then if, as in the figure, GV lie between the intersections, p, q of Pp, Qp, and Pq, Qq respectively, it is always possible to find in the vertical

a point v such that vP, vQ, shall lie within the respective cones of resistance. In fact, in Fig. 1 any point from r to r' will satisfy the condition. Hence, if vg be taken to represent the sum of the weights of P and Q, and if gs, gs' be drawn parallel to Pv, Qv, respectively, the system will be in equilibrium if the resistances at P and Q be represented in magnitude and direction by vs', vs respectively. But from what has been said, Chapter I., it is plain that these lines are possible *directions* for the forces of resistance, and as the *magnitudes* of these forces are unlimited, the proposed representation is possible. If, therefore, the vertical through the centre of gravity of P and Q lie between p and q the position is one of possible equilibrium, and the resistances at P and Q are represented by vs, vs'. Since, moreover, the point v has been assumed, within certain limits, arbitrarily, the resistances at P, Q, are not absolutely determinate.

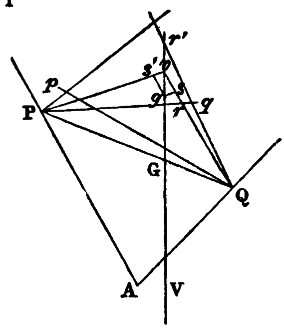

Fig. 1.

If the vertical GV pass *through* either of the points p or q, equilibrium is still possible, but the resistances are no longer indeterminate. Thus, let GV pass through p (Fig. 2). Then it is plain that Pp, Qp, are the only possible lines of direction of the forces of resistance. For if we connect the points P, Q, with any point on GV other than p, one, at least, of the connecting lines will lie outside the cone of resistance. If then we represent by pg the sum of the weights, and draw the parallels gs, gs', the resistances at P and Q will be represented in magnitude and direction by the determinate lines ps, ps'.

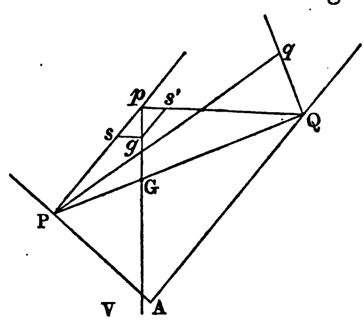

Fig. 2.

If GV lie beyond p or q, it is impossible to connect P and Q with *any* point in the vertical, without causing one, at least, of the connecting lines to fall outside the cone of resistance. In this case the position is not one of equilibrium.

If either pair of the lines Pp, Pq, or Qp, Qq, coincide; in other words, if the vertical plane through the connecting rod *touch* either of the cones of resistance P, it is easy to see that equilibrium is possible, if the vertical GV (Fig. 3) lie between p and q. In this case the only possible directions of resistance are Pv, Qv, and these forces are perfectly determinate.

If the vertical plane through the connecting rod lie wholly outside either cone of resistance, equilibrium is manifestly impossible.

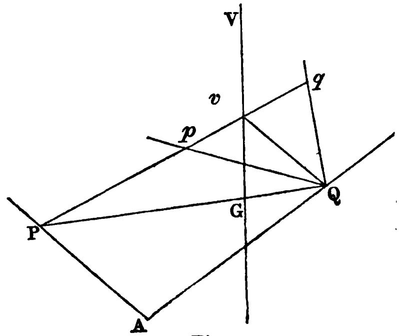

Fig. 3.

5. We learn from this example—1. That a system in which friction is one of the acting forces may have an infinite number of positions of equilibrium lying within certain limits. 2. That the resistance actually developed at each point of the system cannot in general, as in the case of smooth surfaces, be determined simply by the equations of equilibrium, even if the position of the system be given. 3. That in certain cases this indeterminateness disappears, the equations of equilibrium enabling us to assign the actual values to the resistances.

6. As another example, let it be proposed to determine the best angle for draught on a rough inclined plane.

Let W be the weight of the body, P the power applied to the rope, θ the angle of inclination of the power to the inclined plane, and α the inclination of the plane. Then, if the body be just on the point of being dragged up, the equations of equilibrium are

$$P \cos \theta = W \sin \alpha + F, \quad R = W \cos \alpha - P \sin \theta, \quad F = \mu R = R \tan \epsilon ;$$

where R is the pressure on the plane and F the force of friction. Eliminating F and R, we find

$$P \cos (\theta - \epsilon) = W \sin (\alpha + \epsilon). \tag{1}$$

It is evident from this equation that P is a minimum when $\theta = \epsilon$. Hence the best direction of draught makes with the inclined plane an angle equal to the angle of friction. If $\alpha + \epsilon < 90^\circ$, there will always be a gain of power in dragging the body up

the plane, rather than in lifting it vertically. If $a + \epsilon = 90^\circ$, it is easy to see that the direction of the force will be vertical. If $a + \epsilon > 90^\circ$, it might seem that P is still less than W, and that therefore there would be still a gain of power in dragging the weight up the plane. But if in the value of R we make $\theta = \epsilon$ and $P = W \sin(a + \epsilon)$, we shall have

$$R = W \cos(a + \epsilon) \cos \epsilon, \tag{2}$$

which is negative if $a + \epsilon > 90^\circ$. It would be, therefore, impossible to drag the weight up the plane by a force applied in this direction. In fact, it is easy to see that a force so applied will give with the weight a resultant *down*, not *up*, the plane. Hence we infer that if $a + \epsilon < 90^\circ$, it is better to drag the weight, and that if $a + \epsilon > 90^\circ$, it is better to raise it vertically. For, in order that the force and the weight may give a resultant *up* the plane, it is easily seen that we must have $a + \theta < 90^\circ$. Hence, if $a + \epsilon > 90^\circ$, we must have $\theta < \epsilon$.

Also, since $a + \theta < 90^\circ$, we must have

$$a + \epsilon - 90^\circ < \epsilon - \theta;$$

whence $\sin(a + \epsilon) > \cos(\theta - \epsilon)$, and therefore from equation (1) $P > W$, or there is a loss of power in dragging the body up the plane.

We now proceed to consider generally how many indeterminate quantities remain in any case in which the positions of the particles are given.

II.—*Equilibrium of a System of Material Particles.*

Prop. I.

7. A number, n, of material particles, each of which is situated on a rough surface, are connected by certain equations of condition, and are acted on by given forces. Given the position of the system, to determine the conditions necessary to its equilibrium, and the force of resistance actually developed at each point, if it be determinate.

Let $x_1\, y_1\, z_1$, $x_2\, y_2$, z_2, &c., be the co-ordinates of the several particles, and let

$$L = 0,\ M = 0,\ \text{\&c.},$$

be the equations of condition. Let $X_1\ Y_1\ Z_1$ be the components of the external force at the point $x_1\ y_1\ z_1$, and $X'_1\ Y'_1\ Z'_1$ the components of the effective force of resistance at the same point. Then if the particle $x_1\ y_1\ z_1$ be in equilibrium, we have

$$\left.\begin{aligned} X_1 + X'_1 + \lambda \frac{dL}{dx_1} + \mu \frac{dM}{dx_1} + \&c., = 0, \\ Y_1 + Y'_1 + \lambda \frac{dL}{dy_1} + \mu \frac{dM}{dy_1} + \&c., = 0, \\ Z_1 + Z'_1 + \lambda \frac{dL}{dz_1} + \mu \frac{dM}{dz_1} + \&c., = 0. \end{aligned}\right\} \qquad (3)$$

Similar equations will hold for the remaining particles.

Conversely, we know from the principle which has been enunciated above (Art. 3) that the forces of effective resistance will, if possible, take such directions and magnitudes as to keep the system in equilibrium. We have seen, further, that the only restriction upon the resistance of a rough surface is that its direction shall not make with the normal to the supporting surface an angle greater than the angle of friction. Hence, if $a_1\ b_1\ c_1$ be the direction cosines of the normal to the supporting surface, and if we assume

$$R_1^2 = \left(X_1 + \lambda \frac{dL}{dx_1} + \mu \frac{dM}{dx_1} + \&c.\right)^2 + \left(Y_1 + \lambda \frac{dL}{dy_1} + \mu \frac{dM}{dy_1} + \&c.\right)^2 + \left(Z_1 + \lambda \frac{dL}{dz_1} + \mu \frac{dM}{dz_1} + \&c.\right)^2$$

the required condition will be

$$a_1 \left(X_1 + \lambda \frac{dL}{dx_1} + \mu \frac{dM}{dx_1} + \&c.\right) + b_1 \left(Y_1 + \lambda \frac{dL}{dy_1} + \mu \frac{dM}{dy_1} + \&c.\right) + c_1 \left(Z_1 + \lambda \frac{dL}{dz_1} + \mu \frac{dM}{dz_1} + \&c.\right) > R_1 \cos \epsilon_1, \qquad (4)$$

where ϵ_1 is the angle of friction.

Similar conditions hold at the other points of the system. If these conditions, which are n in number, be satisfied, the system is in equilibrium. The components $X'_1\ Y'_1\ Z'_1$, &c., of the seve-

ral forces of resistance are then completely determined by the equations of equilibrium (3) in terms of the quantities λ, μ, &c., which are proportional to the intensities of the geometrical forces, and are in number equal to the equations of condition. There is nothing in the conditions of equilibrium to determine these quantities, nor even to establish any equations among them. They are *restricted* in their values by the conditions (4), but remain, so far as conditions of equilibrium are concerned, indeterminate and independent. The forces of effective resistance at the several points of the system, which are, as we have seen, functions of λ, μ, &c., are, therefore, in general, indeterminate. The entire conclusion may be stated as follows :—

If a number, n, of material particles, each of which rests upon a rough surface, and is acted on by a given force, be connected by m equations of condition, and if the entire system be in equilibrium, in a given position, the values of the forces of effective resistance, and of the geometrical forces, will in general contain m indeterminate quantities.

It must be observed in applying this principle, that the equations of the supporting surfaces are not reckoned among the equations of condition.

8. If any one or more of the particles be on the point of slipping, the number of the indeterminate quantities is reduced. If, for example, the condition of the system be such that the point $x_1\ y_1\ z_1$ would slip if its supporting surface were rendered in any degree smoother, the friction developed at this point would have its greatest value, and consequently the line of effective resistance would be most oblique to the normal. The line of effective resistance, therefore, would lie *upon*, not *within*, the cone of resistance, and we should thus have an additional equation. The number of indeterminate quantities is therefore diminished by one for each particle which is in such a position, which may be termed an *extreme* position of equilibrium.*

Thus, for example, in the problem discussed above (Art. 4), there is one geometrical condition, and consequently, in general, one indeterminate quantity in the solution. But we have seen

* *Vide* Chap. III.

that if the vertical plane through the connecting rod touch either of the cones of resistance, or if the vertical through the centre of gravity pass through either of the points p or q, the solution becomes determinate. This is in accordance with the general principle just enunciated.

9. Before proceeding further, we must say a few words in explanation of the *indeterminateness* which appears in the solution of this and similar problems. When we find a mechanical problem to be indeterminate, the inference to be drawn is this, that the *data* are insufficient. In the present case we shall see that the conditions of the problem are not complete unless the values of the forces which we have termed "geometrical" be given. This will appear most readily by considering these forces as they really are, namely, the limiting values of the forces of elasticity which are produced by the extension or contraction of the bars or laminæ by which it is attempted to realize mechanically the geometrical conditions.* If the amount of this extension or contraction be supposed to be indefinitely diminished, the corresponding force of elasticity remaining finite, the limit thus attained is the force which we have termed "geometrical." Now, while the extension or contraction continues finite, it is plain that the problem will not necessarily be determinate, unless the elastic force, which is developed by the extension or contraction, be given. If this force be given, as well as the positions of the several particles, each particle may be considered as isolated from the others and acted on by a system of forces which are known. The equations of equilibrium will then be sufficient (and no more than sufficient) to determine the unknown quantities, namely, the resistance of the supporting surface in magnitude and direction. But if the elastic force be not given, the particle is acted on by a force of unknown intensity, and therefore the equations of equilibrium are insufficient to determine all the unknown quantities. The same will necessarily hold when we pass to the limit, and thus substitute the geometrical force for the force of elasticity. Unless the intensity of this force be given, the *data* of the problem are insufficient. In

* Chap. I., § 4.

fact, the intensity of this force might be varied more or less largely without disturbing the equilibrium.

Thus, for example, if the system consist of a single particle which is subject to the geometrical condition that its distance from a fixed point shall be invariable; this condition is mechanically realized by connecting the particle with the fixed point by a rigid rod.

If this rod were perfectly inextensible, the condition would be perfectly realized. The geometrical or equivalent force would then be a force directed along the rod and absolutely undetermined in magnitude. But we know that there is in nature no such thing as an inextensible rod, and that the force which is really introduced into the system by such a connection is developed by the actual extension or contraction of the rod. The true conception, therefore, of the force which we have termed geometrical is, that it is *the limit to which the elastic force developed by the extension or contraction of the rod tends, when the coefficient of elasticity is indefinitely increased.* The *data* of the problem into which such a force enters are not then necessarily sufficient, unless we know the actual amount of the extension or contraction of the rod. This extension or contraction is not necessarily determined by the conditions of equilibrium. For it might happen that the equilibrium would not be disturbed by replacing the connecting rod by another in which the extension or contraction should be quite different. If, therefore, we have no other information than that afforded by conditions of equilibrium, we must in such a case look upon the force of elasticity as indeterminate. This conclusion being independent of the actual value of the coefficient of elasticity, will necessarily hold also for the geometrical or replacing force, which is, as we have seen, the limit of the force of elasticity. This force cannot, therefore, in general be determined by the equations of equilibrium.

Prop. II.

10. Let a system of material particles be connected as in Prop. I., and let it be supposed that the equations (3) are satis-

fied. To determine what further conditions are necessary for the equilibrium of the system.

We have seen (p. 17) that for each particle of which the system is composed we have the condition

$$a_1\left(X_1+\lambda\frac{dL}{dx_1}+\mu\frac{dM}{dx_1}+\&c.\right)+b_1\left(Y_1+\lambda\frac{dL}{dy_1}+\mu\frac{dM}{dy_1}+\&c.\right)$$
$$+c_1\left(Z_1+\lambda\frac{dL}{dz_1}+\mu\frac{dM}{dx_1}+\&c.\right)>R_1\cos\epsilon_1,$$

retaining the notation of Prop. I. Hence if we assume

$$\phi_1=\left\{a_1\left(X_1+\lambda\frac{dL}{dx_1}+\&c.\right)+b_1\left(Y_1+\lambda\frac{dL}{dy_1}+\&c.\right)\right.$$
$$\left.+c_1\left(Z_1+\lambda\frac{dL}{dz_1}+\&c.\right)\right\}^2-\left\{\left(X_1+\lambda\frac{dL}{dx_1}+\&c.\right)^2\right.$$
$$\left.+\left(Y_1+\lambda\frac{dL}{dy_1}+\&c.\right)^2+\left(Z_1+\lambda\frac{dL}{dz_1}+\&c.\right)^2\right\}\cos^2\epsilon_1;$$
$$\phi_2=\{a_2(X_2+\&c.)+\&c.\}^2-\{(X_2+\&c.)^2+\&c.\}\cos^2\epsilon_2;$$
$$\phi_3=\&c.;$$

the required conditions will be

$$\phi_1>0,\quad \phi_2>0,\ \ldots\ldots\ \phi_n>0, \tag{4}$$

each one of the quantities ϕ_1, ϕ_2, &c., being a function of one or more of the coefficients λ, μ, &c., which, as we have seen in Prop. I., remain completely undetermined by the equations of equilibrium (3). We have then to inquire whether it be possible to assign to these coefficients such values as will satisfy these conditions.

Let λ, μ, ν, &c., be a system of values satisfying the conditions

$$\phi_1>0,\ .\ \phi_2>0,\ \ldots\ldots\ \phi_n>0.$$

Let it be supposed that λ, μ, ν, &c., are varied continuously until we arrive at a system of values which causes one of these functions to vanish. Let ϕ_1 be the vanishing function. Then the

system of values of λ, μ, ν, &c., at which we have arrived, is such as to satisfy the conditions

$$\phi_1 = 0, \quad \phi_2 > 0, \; \ldots\ldots \; \phi_n > 0.$$

Restricting now the values of λ, μ, ν, &c., to such as satisfy the condition $\phi_1 = 0$, let these quantities be again continuously varied. Then, unless the relation established by $\phi_1 = 0$ be such as to make all the other functions ϕ_2, ϕ_3, &c., essentially positive, we shall at last arrive at a system of values which will cause a second function, ϕ_2, to vanish. This system satisfies the conditions

$$\phi_1 = 0, \quad \phi_2 = 0, \quad \phi_3 > 0, \; \ldots\ldots \; \phi_n > 0.$$

Restricting now the values of λ, μ, ν, &c., to those which satisfy the conditions $\phi_1 = 0$, $\phi_2 = 0$, let them be varied as before. Then, unless the remaining functions are essentially positive, we shall, as before, arrive at a system which causes a third function, ϕ_3, to vanish. Continuing this process, we shall at length arrive at a system which causes a certain number of the functions to vanish, these functions satisfying the condition that every system which causes them to vanish renders the remaining functions positive or zero. Conversely, if the conditions (4) be fulfilled, it will always be possible to find a number, p, of the functions ϕ, &c., such that if the equations

$$\phi_1 = 0, \quad \phi_2 = 0, \; \ldots\ldots \; \phi_p = 0 \tag{5}$$

be satisfied, the remaining functions ϕ_{p+1}, ϕ_{p+2} . . . . . ϕ_n shall be essentially positive. If $p = m$, the foregoing equations are sufficient to determine the quantities λ, μ, &c. In this case the remaining functions, ϕ_{p+1}, &c., will be constants depending upon the acting forces and the positions of the particles. The conditions of equilibrium expressed in terms of the given quantities of the problem will therefore be

$$\phi_{p+1} > 0, \quad \phi_{p+2} > 0, \; \ldots\ldots \; \phi_n > 0,$$

besides a number of conditions denoting that the equations (5) shall have at least one system of real roots.

If $p < m$ the quantities λ, μ, &c., are not completely determined, and the remaining functions, ϕ_{p+1}, &c., will contain a certain number of these quantities. The conditions (5) will then be fulfilled independently of them.

11. To take a simple case of the general proposition, let it be supposed that the system is composed of two particles connected by a single geometrical equation. We shall have then

$$\phi_1 = A_1\lambda^2 + 2B_1\lambda + C_1 \quad \phi^2 = A_2\lambda^2 + 2B_2\lambda + C_2,$$

where

$$A_1 = \left(a_1\frac{dL}{dx_1} + b_1\frac{dL}{dy_1} + c_1\frac{dL}{dz_1}\right)^2 - \left(\frac{dL^2}{dx_1^2} + \frac{dL^2}{dy_1^2} + \frac{dL^2}{dz_1^2}\right)\cos^2\varepsilon_1,$$

$$B_1 = \left(a_1X_1 + b_1Y_1 + c_1Z_1\right)\left(a_1\frac{dL}{dx_1} + b_1\frac{dL}{dy_1} + c_1\frac{dL}{dz_1}\right)$$

$$- \left(X_1\frac{dL}{dx_1} + Y_1\frac{dL}{dy_1} + Z_1\frac{dL}{dz_1}\right)\cos^2\varepsilon_1,$$

$$C_1 = (a_1X_1 + b_1Y_1 + c_1Z_1)^2 - (X_1^2 + Y_1^2 + Z_1^2)\cos^2\varepsilon_1,$$

with similar values for A_2, B_2, C_2. Then

(1). If $A_1 > 0$, $A_2 > 0$, the conditions $\phi_1 > 0$, $\phi_2 > 0$ can always be satisfied by taking λ sufficiently large.

(2). Let $A_1 < 0$, $A_2 < 0$. Then unless $B_1^2 - A_1C_1 > 0$, $B_2^2 - A_2C_2 > 0$, the conditions $\phi_1 > 0$, $\phi_2 > 0$, cannot be fulfilled by any value of λ whatever. If $B_1^2 - A_1C_1 > 0$, $B_2^2 - A_2C_2 > 0$, let the roots of the equations $\phi_1 = 0$, $\phi_2 = 0$ be λ_1, λ'_1, and λ_2, λ'_2 respectively. Then since ϕ_1, ϕ_2, are evidently negative for $\lambda = \pm\infty$, the conditions $\phi_1 > 0$, $\phi_2 > 0$ can only be satisfied by values lying between λ_1 and λ'_1, and also between λ_2 and λ'_2. Hence if we suppose that $\lambda_1 < \lambda'_1$ and $\lambda_2 < \lambda'_2$, the required conditions will be $\lambda_1 < \lambda'_2$ and $\lambda_2 < \lambda'_1$.

(3). Let $A_1 > 0$, $A_2 < 0$. Then it is evident that the required value of λ must, as before, lie between the roots $\lambda_2\,\lambda'_2$, and must not lie between the roots $\lambda_1\,\lambda'_1$; we must have therefore either $\lambda'_2 < \lambda_1$ or $\lambda_2 > \lambda'_1$. Similar conclusions hold for the case $A_1 < 0$, $A_2 > 0$.

(4). If $C_1 > 0$, $C_2 > 0$ the conditions $\phi_1 > 0$ $\phi_2 > 0$ can always be satisfied by making $\lambda = 0$.

12. To interpret these several cases geometrically, let the normal to the supporting surface at either point $x_1\, y_1\, z_1$, the direction of the applied force R_1, and the direction of the geometrical force be projected upon a sphere by radii parallel to these lines. Let these radii meet the sphere in the points N, R, L respectively (Fig. 4).

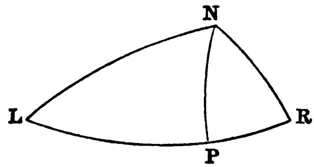

Fig. 4.

Then putting

$$V_1^2 = \frac{dL^2}{dx_1^2} + \frac{dL^2}{dy_1^2} + \frac{dL^2}{dz_1^2}$$

we have, as is easily seen,

$$A_1 = V_1^2 (\cos^2 LN - \cos^2 \epsilon_1), \quad C_1 = R_1^2 (\cos^2 RN - \cos^2 \epsilon_1),$$

$$B_1 = V_1 R_1 (\cos LN \cos RN - \cos LR \cos^2 \epsilon_1).$$

The condition $A_1 > 0$ is therefore equivalent to $LN < \epsilon_1$, denoting that at the point $x_1\, y_1\, z_1$ the line of direction of the geometrical force lies within the cone of resistance.

The condition $C_1 > 0$ is equivalent to $RN < \epsilon_1$, denoting that at the point $x_1\, y_1\, z_1$ the line of direction of the applied force lies within the cone of resistance.

If, therefore, as in (1), $A_1 > 0$ $A_2 > 0$, denoting that at each point the line of direction of the geometrical force lies within the cone of resistance, it is plain that the intensity of this force may always be taken sufficiently great to bring the lines of direction of the complete resultants within the cones of resistance. This is, as we know, sufficient for equilibrium.

If, as in (4), $C_1 > 0$ $C_2 > 0$, denoting that at each point the line of direction of the applied force lies within the cone of resistance, it is plain that the intensity of the geometrical force may be taken sufficiently small to bring the complete resultants within the cones of resistance, and thus to secure equilibrium. In fact, in this case each particle would be in equilibrium if separated from the other.

To interpret the condition

$$B_1^2 - A_1C_1 > 0,$$

we have from the foregoing values of A_1 B_1 C_1

$$B_1^2 - A_1C_1 = V_1^2R_1^2 \cos^2\epsilon_1 (\cos^2 LN + \cos^2 RN \\ - 2\cos LN \cos RN \cos LR - \sin^2 LR \cos^2\epsilon_1).$$

But if NP be perpendicular to LR, it is easily shown that

$$\cos^2 LN + \cos^2 RN - 2 \cos LN \cos RN \cos LR = \sin^2 LR \cos^2 NP.$$

Hence we have

$$B_1^2 - A_1C_1 = V_1^2R_1^2 \cos^2\epsilon_1 \sin^2 LR (\cos^2 NP - \cos^2\epsilon_1).$$

The condition $B_1^2 - A_1C_1 > 0$ is, therefore, equivalent to $NP < \epsilon_1$. But it is easily seen that this condition is necessary to equilibrium. For the arc NP measures the inclination of the normal to the plane containing the directions of the external force and the geometrical force, and therefore the *minimum* angle which a line situated in this plane can make with the normal. But the line of direction of the total force acting upon the particle, being the resultant of the external and geometrical forces, is necessarily situated in this plane. If, therefore, NP were greater than ϵ_1, the total acting force would necessarily make with the normal an angle greater than ϵ_1; equilibrium would, therefore, be impossible. It is evident that if either A_1 or C_1 be positive, $B_1^2 - A_1 C_1$ will also be positive. For if either LN or RN be less than ϵ_1, NP must necessarily be less than ϵ_1.

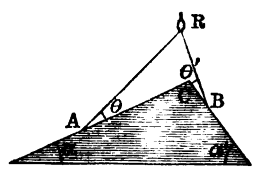

Fig. 5.

13. As an example of this principle, we shall consider the case of two material particles, A, B (Fig. 5), resting upon two rough inclined planes, and connected by a string passing through a fixed smooth ring.

Let W, W' be the weights of the particles, T the tension of the string, and ϵ, ϵ' the angles of friction of the planes. Let also F, F' be the effective forces of friction reckoned upwards,

and N, N' the normal reactions. Then the equations of equilibrium are

$$T\cos\theta - W\sin\alpha + F = 0, \quad T\cos\theta' - W'\sin\alpha' + F' = 0,$$

$$T\sin\theta - W\cos\alpha + N = 0, \quad T\sin\theta' - W'\cos\alpha' + N' = 0.$$

It is further necessary to equilibrium that F, F' respectively should lie within the limits

$$F = \pm N\tan\epsilon. \quad F' = \pm N'\tan\epsilon'.$$

Substituting the values of F, F', N, N' from the equations of equilibrium, we have at the limits

$$T = W\frac{\sin(\alpha\mp\epsilon)}{\cos(\theta\pm\epsilon)}, \qquad T = W'\frac{\sin(\alpha'\mp\epsilon')}{\cos(\theta\pm\epsilon')}.$$

Let $\alpha > \epsilon$, $\alpha' > \epsilon'$. Then it is easily seen that the least value of T in each case will correspond to the upper sign.* Hence, according to the principle stated (p. 21), equilibrium will be possible if either of the following conditions be fulfilled.

$$\frac{W\sin(\alpha+\epsilon)}{\cos(\theta-\epsilon)} \text{ not } < \frac{W'\sin(\alpha'-\epsilon')}{\cos(\theta'+\epsilon')},$$

$$\frac{W'\sin(\alpha'+\epsilon')}{\cos(\theta'-\epsilon')} \text{ not } < \frac{W\sin(\alpha-\epsilon)}{\cos(\theta+\epsilon)}.$$

Hence it is evident that the ratio of the weights must lie within the limits

$$\frac{\sin(\alpha'-\epsilon')}{\sin(\alpha+\epsilon)}\,\frac{\cos(\theta-\epsilon)}{\cos(\theta'+\epsilon')}, \qquad \frac{\sin(\alpha'+\epsilon')}{\sin(\alpha-\epsilon)}\,\frac{\cos(\theta+\epsilon)}{\cos(\theta'-\epsilon')}.$$

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* Unless

$$\alpha+\theta > \frac{\pi}{2} \text{ or } \alpha'+\theta' > \frac{\pi}{2},$$

in which case both parts of the string lie at the same side of the vertical. It is easy to modify the reasoning so as to suit this case, as also the case in which either α or α' is less than ϵ.

III.—*Equilibrium of a System of Material Particles each of which is constrained to remain upon a Rough Curve.*

14. A slight modification of the geometrical construction of p. 8 will be sufficient to adapt to the case of rough curves the reasoning which has been applied to the case of rough surfaces. In the case of surfaces, the normal reaction has a definite direction, while the direction of the force of friction is indefinite. In the case of curves, on the other hand, the normal reaction is indefinite in its direction, which may lie anywhere in the normal plane, while the force of friction must be directed in the tangent to the curve, which is a definite line. The geometrical condition by which the resistance of a rough curve is limited is therefore evidently this: that its direction shall not make with the tangent an angle less than the complement of the angle of friction. The geometrical construction, therefore, by which we replace that of p. 8 is this:—

Let a cone of revolution be described round the tangent to the curve at any point, the semi-angle of the cone being the complement of the angle of friction. Then it is evident by the same reasoning as before that the resistance of a rough curve is limited by one only condition, namely, that its line of direction shall not lie *within* this cone. With this modification the methods of Prop. I. and II. are strictly applicable.

Thus the equations of equilibrium (3) are true here without any change in their form. The condition (4) applied to this case may be written thus:

$$a_1\left(X_1 + \lambda\frac{dL}{dx_1} + \text{\&c.}\right) + b_1\left(Y_1 + \lambda\frac{dL}{dy_1} + \text{\&c.}\right)$$
$$+ c_1\left(Z_1 + \lambda\frac{dL}{dz_1} + \text{\&c.}\right) < R_1 \cos\varepsilon_1,$$

where $a_1\, b_1\, c_1$ are the direction cosines of the tangent to the curve, the other letters having the same signification as in Prop. I. The rest of the discussion is the same as before.

IV.—*Equilibrium of a Solid Body supported by one or more Rough Surfaces.*

15. The principles upon which this problem is discussed are identical with those already laid down. At each point of support a force acts, whose intensity is unlimited, and whose line of direction is subject to one condition only, namely, that it shall not lie outside a given cone. In order, therefore, to determine the conditions of equilibrium, we must consider whether it be possible, consistently with this limitation, to assign to these forces such magnitudes and directions as shall keep the body at rest.

Before considering the question generally, we shall examine the following particular case:—

Prop. II.

A solid body of any form is supported by two fixed rough surfaces, and is acted on by the force of gravity alone. To determine the conditions of its equilibrium, and the resistance developed at each point of contact, if it be determinate.

In this case the body is kept in equilibrium by three forces, namely, the weight, acting at the centre of gravity, and the two resistances, acting respectively at the two points of support. In accordance with a well-known principle, the lines of direction of these three forces must lie in the same plane, and pass through the same point. Hence it is evident that the plane containing the centre of gravity and the two points of support is necessarily a vertical plane, inasmuch as it contains the line of direction of the force of gravity. Let this plane be represented by the plane of the paper, the shaded parts of the figure representing the sections of the body and the two supporting surfaces. Let G (Fig. 6) be the centre of gravity, and GV a vertical. The rest of the discussion of the pre-

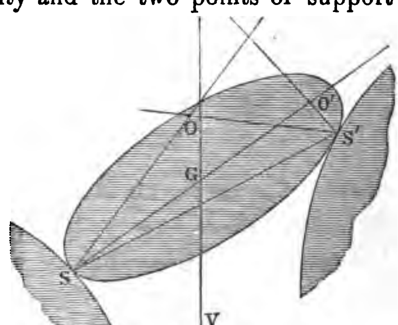

Fig 6.

sent question is similar to that of p. 14. If SO, SO', $S'O$, $S'O'$ be the sections of the cones of resistance, equilibrium is possible, if GV lie between O and O', the effective resistances at S and S' being in this case indeterminate. It is also possible, if GV pass through O or O', the resistances being then determinate. All the other conclusions of Art. 4 hold equally here. We now proceed to consider the general case of a body supported by two surfaces.

Prop. III.

16. A solid body of any form, supported by two fixed rough surfaces, is acted on by forces of any kind. To determine the conditions of its equilibrium, and the resistance developed at each point of contact, if it be determinate.

Let $X\ Y\ Z$ be the components of the force acting at any point of the supported body, and assume

$$X_1 = \int X dm, \quad Y_1 = \int Y dm, \quad Z_1 = \int Z dm,$$

and

$$L_1 = \int (zY - yZ)\, dm, \; M_1 = \int (xZ - zX)\, dm, \; N'^1 = \int (yX - xY)\, dm.$$

Let R, R' be the effective resistances at the points of support, and $\alpha\, \beta\, \gamma$, $\alpha'\, \beta'\, \gamma'$ the cosines of the angles which their directions make with the co-ordinate axes. Then if $x\ y\ z$, $x'\ y'\ z'$ be the co-ordinates of the points of support, the equations of equilibrium will be

$$X_1 + \alpha R + \alpha' R' = 0, \quad Y_1 + \beta R + \beta' R' = 0, \quad Z_1 + \gamma R + \gamma' R' = 0,$$

$$L_1 + R(\beta z - \gamma y) + R'(\beta' z' - \gamma' y') = 0,$$

$$M_1 + R(\gamma x - \alpha z) + R'(\gamma' x' - \alpha' z') = 0, \qquad (6)$$

$$N_1 + R(\alpha y - \beta x) + R'(\alpha' y' - \beta' x') = 0.$$

If the position of the body be given, these equations contain six unknown quantities, namely, R, R', and four of the six cosines. It would seem, therefore, as if there were sufficient equations to determine the forces of resistance both in magnitude and direction. We shall show, however, that there are in reality but five independent equations between these unknown quantities, the sixth being a relation between the given quantities of the prob-

lem, necessary to the equilibrium of the body. For if the equations of moments be multiplied respectively by $x - x'$, $y - y'$, $z - z'$, and added, we have

$$L_1(x-x') + M_1(y-y') + N_1(z-z'),$$
$$+ (xy' - x'y)(R\gamma + R'\gamma') + (zx' - xz')(R\beta + R'\beta')$$
$$+ (yz' - zy')(R\alpha + R'\alpha') = 0;$$

or, substituting from the equations of forces for R and R'

$$L_1(x-x') + M_1(y-y') + N_1(z-z')$$
$$= (xy' - yx')\,Z_1 + (zx' - xz')\,Y_1 + (yz' - zy')\,X_1.$$

From this equation, which is wholly independent of the forces of resistance, it is easy to see that if the external forces acting upon the body be resolved into a single force passing through any point in the line SS' (Fig. 4), and a moment with regard to this point, the plane of this moment must pass through the line SS'. This conclusion is evident in itself. Neither of the forces R, R' has a moment round the line SS'; if, therefore, the acting forces had a moment round this line, the system could not be in equilibrium.

Thus, for example, in the problem of Art. 15, we have seen that the vertical plane through the centre of gravity, G, must pass through the line SS'. This might be otherwise expressed by saying that the vertical line through G must intersect SS'; or, in other words, that the force of gravity has no moment round this line.

We learn from this discussion that inasmuch as there are but five equations among the six quantities required to determine the forces of resistance, one of these quantities must remain indeterminate. If, however, any additional condition be given, as, for example, if it be given that at either of the points S, S', the force of friction has its extreme value, the problem becomes determinate.

It is easy to see that each of the forces R, R' must lie in a certain determinate plane. This may be shown by eliminating $\alpha'\,\beta'\,\gamma'\,R\,R'$ between the equations (6), when it will be found that the result is an equation of the form

$$A\alpha + B\beta + C\gamma = 0. \qquad (8)$$

Similarly for $\alpha' \beta' \gamma'$ we should have

$$A'\alpha' + B'\beta' + C\gamma' = 0. \qquad (9)$$

But it may be proved geometrically as follows:—Let S' be taken as the centre of moments. Then since R' has no moment with regard to S', it is plain that the resultant moment of the applied forces with regard to this point must be equal and opposite to the moment of R with regard to the same point. Hence it is evident that R must lie in the plane of this resultant moment. Similarly R' must lie in the plane of the resultant moment of the applied forces with regard to the point S. It is evident from p. 30 that both these planes pass through SS'.

A third equation between $\alpha \beta \gamma$, $\alpha' \beta' \gamma'$, will be obtained by eliminating R and R' between the first three equations (6). We have then

$$(\beta\gamma' - \gamma\beta')\, X_1 + (\gamma\alpha' - \alpha\gamma')\, Y_1 + (\alpha\beta' - \beta\alpha')\, Z_1 = 0. \qquad (10)$$

This equation denotes that R, R', and the resultant force are all parallel to the same plane. This is evident in itself, inasmuch as these three forces ought to equilibrate when transferred to the same point.

It remains to consider whether any relation among the given quantities of the problem can be derived from the condition, that neither of the forces R, R' can make with the corresponding normal an angle exceeding the angle of friction for that surface. This relation may be investigated as follows:—

Let a sphere be described round any arbitrary point, and let the several lines and planes of the problem be projected on this sphere by parallel lines and planes through its centre.

Let S (Fig. 7) be the projection of SS', and let SM, SM', be the projections of the planes of the resultant moments of the applied forces, taken respectively with regard to the centres S' and S''. Let also F be the projection of the resultant force, and N,N' the projections of the normals to the supporting surfaces measured in each case outwards.

Then if R,R' be the projections of the forces R,R', we know

that R lies in the circle SM' and R' in the circle SM. Also if ϵ, ϵ' be the angles of friction, we have

$$RN \text{ not} > \epsilon, \quad R'N' \text{ not} > \epsilon'.$$

Hence if $\epsilon < NP$ or $\epsilon' < N'P'$ equilibrium is impossible. If neither of these inequalities be true, let

$$NE = NE_1 = \epsilon, \quad \text{and } N'E' = N'E'_1 = \epsilon'.$$

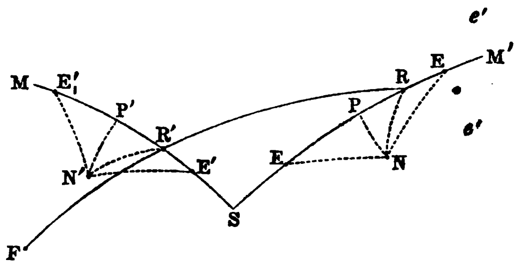

Fig. 7.

Then R, R' must be situated respectively in the arcs EE_1, $E'E'_1$. Moreover, it is plain from equation (10) that R', R, F lie in the same great circle. Hence, in order that there may be equilibrium, it must be possible to draw through F a great circle cutting the two arcs EE_1, $E'E'_1$.

The discussion of the special cases of this problem is analogous to that of Art. 4. If we draw the circles FE, FE_1, and if these circles intercept a finite part of the arc $E'E'_1$, the friction actually developed at the points of contact is indeterminate. If the points F, E'_1, E, or F, E', E_1, lie in a great circle, the system is in an extreme position of equilibrium, and the actually developed friction is determinate.

We shall now give some examples of the general problem.

Example 1.

17. A circular cylinder rests in a horizontal position upon a rough inclined plane, and is supported by a string coiled round its middle section, and supporting a weight. To determine the conditions of equilibrium.

Let the plane of the paper (Fig. 8) be that of the middle section. Then the cylinder is acted on by three forces; namely, 1. The attached weight, W', acting in the vertical line HW'; 2. The weight of the cylinder itself, W, acting in the vertical line CV; 3. The reaction of the rough plane R, acting at the point T. It is evidently necessary for equilibrium that this force should also be vertical. The angle which R makes with the perpendicular to the plane is therefore TCV (= inclination of plane). Hence it is evident that the inclination of the plane must not exceed the angle of friction. Moreover, the resultant of W and W' must pass through T. Hence it is easy to see that, if a be the inclination of the plane,

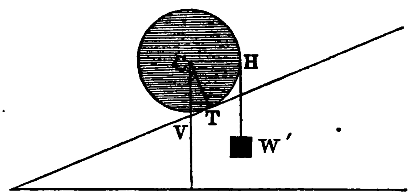

Fig. 8.

$$W \sin a = W' (1 - \sin a).$$

If these two conditions be fulfilled, the system will be at rest.

We may inquire, further, 1. Whether the cylinder can ascend or descend without revolving; 2. Whether it can revolve without ascending or descending.

1. It is plain that the cylinder cannot ascend without revolving, inasmuch as such a movement would raise the centre of gravity of the whole system. In order that it may descend without revolving, the accelerating force down the plane must exceed the *maximum* force of friction, and the moment of all the forces round the centre of gravity must vanish. Hence, evidently,

$$a > \epsilon, \quad \text{and } \mu (W + W') \cos a = W'.$$

2. If the cylinder revolve without ascending or descending, the accelerating force on the centre of gravity must vanish, and the attached weight must exceed the maximum force of friction. Hence, as in the case of complete equilibrium, the reaction, R, must act in the vertical. Since, moreover, the force of friction has its maximum value, R must make with CT an angle equal to the angle of friction. Hence the required conditions are

$$a = \epsilon, \quad (W + W') \sin \epsilon < W'.$$

The reaction, which is in all these cases determinate, is given here, as in the case of complete equilibrium, by the equation,

$$R = W + W'.$$

Example 2.

18. A cylinder is kept in equilibrium, as in Ex. 1, except that the sustaining cord passes over a fixed pulley. To find the conditions of equilibrium.

The acting forces in this case are—1. The weight of the cylinder acting in the vertical CV; 2. The attached weight acting in the line VP; 3. The resistance of the plane acting at T, which must, for equilibrium, be directed in the line VT. We must have, therefore, as one condition,

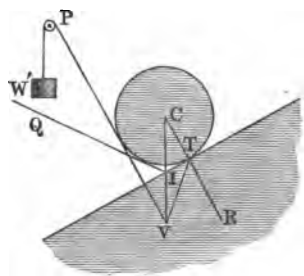

Fig. 9.

$$VTR \text{ not} > \epsilon.$$

If this condition be fulfilled, the attached weight will be determined by the equation,

$$W' \times \sin TVP = \text{weight of cylinder} \times \sin TVC.$$

There is another condition necessary to equilibrium, which is independent both of the attached weight and of the coefficient of friction. The point of intersection V must not be situated *above* I. For, if it were, it would be impossible that the reaction R should equilibrate the resultant of the two weights, inasmuch as its direction would in that case lie in the *salient* angle between the directions of these forces. Hence IQ, making with the horizon an angle equal to the inclination of the plane, is the limit of the possible positions of the cord. It is easy to see that this limit can only be attained with an infinite coefficient of friction.

If the cord, instead of sustaining a weight, were attached to a fixed point, the conditions would be reduced to two; namely, that VTR should not exceed the angle of friction, and that the cord should make with the horizon an angle greater than the inclination of the plane. The foregoing equation would in this case serve to determine the tension.

The reaction of the rough plane, which is determinate in this case also, is given by the equation,

$$R \sin TVP = \text{weight of cylinder} \times \sin CVP.$$

19. If two bodies be in contact, not at a mathematical point, but throughout a finite portion of surface, we must suppose the force of friction to act at each point of the surface of contact. But in such a case it is known that the abstractions of Rational Mechanics do not enable us to determine the actual pressure at each point, even if the surface be smooth. This pressure is really governed by some unknown law depending upon the acting forces, the form of the surface of contact, and the molecular constitution of the bodies. In the investigation of the laws of equilibrium or motion, however, we are in general, in the case of smooth surfaces, enabled to evade this difficulty. Thus, for example, when a heavy body rests upon a smooth horizontal base, we say that there will be equilibrium, if the vertical through the centre of gravity fall within the contour of the base. The validity of this conclusion does not in anywise depend upon the law regulating the pressure at each point. For we know that the action of gravity upon a rigid body is equivalent to a single vertical force passing through the centre of gravity. If this line fall within the contour of the base, the single force may, in the first place, be transferred to the point in which its direction intersects the plane of the base. This force is necessarily equilibrated by the reaction of the plane; and the same will be true if this single force be replaced by a pressure acting at each point of the base. Whatever be the law by which its intensity is governed, provided that its resultant passes through the projection of the centre of gravity, the pressure at each point of the base will be equilibrated by the reaction of the base, and therefore the whole system will be in equilibrium.

20. We may arrive at the same conclusion by another method, which we shall find useful in our present subject. If the system be not in equilibrium, it must begin to move. Now, if the above-mentioned condition be fulfilled, it is easily seen that there can be no movement. For inasmuch as the whole force is vertical,

there can be no horizontal movement; and inasmuch as the re-reaction of the fixed plane is a *resisting* force, there can be no upward movement separating the base from the supporting plane. The only possible movement, then, is a movement of rotation, which is necessarily round a tangent to the contour of the base. But if we assume that such a movement has commenced, the contact of the body with the supporting plane will be at once limited to a single point through which the axis passes. The only force, then, which has a moment round this axis is the weight of the body, which, if its line of direction fall within the contour of the base, will necessarily tend to *destroy* the movement which we have supposed to have commenced. As this destruction must be absolutely simultaneous with the commencement of the movement, it is plain that no such movement is possible.

The same principle may, in certain cases, be applied when friction is one of the acting forces. We must consider, in the first place, what movements are geometrically possible, and then seek to determine what conditions are requisite, in order that the tendency of the forces, which would exist if any one of these movements had actually commenced, may be to destroy, or, at least, not to augment it. If these conditions be fulfilled, there is necessarily equilibrium. If they be not fulfilled, we cannot always say that equilibrium does not exist. For if any of the forces which act upon the body be geometrical, and if it be possible to assign to their intensities such values as would produce equilibrium, then equilibrium is possible, even though these values be different from any which could exist if the system were in motion. We shall consider this point more fully when we come to treat of the difference between necessary and possible equilibrium.

21. As an example of the principle here stated, we may consider the common problem of a heavy body resting upon a rough inclined plane. In this case, the base upon which the body rests being supposed finite, there are two kinds of movement possible; namely, either a rotation of the body round a tangent to the contour of the base, or a slipping of the base itself upon the supporting plane. The former of these movements becomes im-

possible, as in the case of a smooth body, if the vertical through the centre of gravity fall within the base. With regard to the movement of slipping, we know that this movement cannot take place at any point where the direction of the acting force falls within the cone of resistance. If then, as in the preceding case, we conceive the weight of the body to be transferred to the point where the vertical through the centre of gravity intersects the base, and then resolved into a system of parallel forces acting at every point of the base, each of these forces will evidently make the same angle with the perpendicular to the supporting plane. Hence, *whatever be the law which governs the intensity of these forces*, the condition of equilibrium, necessary and sufficient, is, that their common direction (i. e. the vertical) shall not make with the normal to the supporting plane an angle greater than the angle of friction. This is the same as the condition for a single particle.

22. The same principle is applicable to any case in which the forces acting on the body may be replaced by a single force. The conditions of equilibrium in such a case are—that this unique resultant shall pass through some point in the plane of contact, and shall not make with the normal to this plane an angle greater than the angle of friction; the former of these conditions excluding a motion of rotation, and the second excluding a motion of slipping.

As an example of this principle, we may consider the case of a cube resting upon a rough inclined plane, four of its edges being horizontal. Let it be supposed that it is supported by a string attached to the middle point of the upper edge (to which it is perpendicular), passing over a pulley, and sustaining a weight. Let it be required to determine the least coefficient of friction which is consistent with equilibrium.

In this case it is evident that the two acting forces—namely, the weight of the cube, and the tension of the string—lie in the same plane passing through the centre of the cube, and perpendicular to its horizontal edges. These forces have, therefore, a single resultant.

Let the plane of the paper (Fig. 10) represent the plane of the two forces. Then if, as in the case of the cube $A\,B\,C\,D$, a

vertical through the centre of gravity intersects the line $P\,C$ *beyond* C, it is always possible to determine the ratio of W to the weight of the cube such that the resultant of these forces may be perpendicular to the inclined plane. Hence, however small the coefficient of friction may be, equilibrium is always possible. If, on the other hand, as in the case of the cube $A'\,B'\,C'\,D'$, the vertical through the centre intersect $P'B'$ at a point *between* P' and B', it is evident that of all directions of the resultant intersecting the base $A'\,D'$, that which is least oblique to the inclined plane passes through A'. The *minimum* coefficient of friction, therefore, which is consistent with equilibrium is represented by the tangent of the inclination of this line to $A'\,B'$.

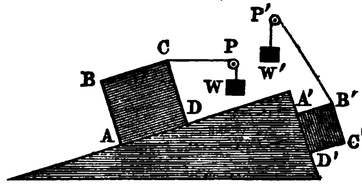

Fig. 10.

23. The power which we have, in these cases, of evading the difficulty arising from the fact that the body rests upon more than three points, depends, as we have seen, upon the constancy of the inclination of the acting force at each point to the supporting surface. The same principle, however, may be extended to the case in which the acting force, though not making a constant angle with the supporting surface itself, does make a constant angle with that curve on the surface in which the corresponding point of the body must necessarily move.

Thus, for example, if a solid homogeneous cylinder, acted on only by gravity, rest within a fixed hollow cylinder of the same radius, the whole system being inclined to the horizon, the force at each point is manifestly indeterminate, and the vertical is inclined to the supporting surface at an angle which assumes all values from 0 to 90°. But it is plain, from considerations of symmetry, that the only movement which such a system could take will be at every point parallel to the axis of the cylinder. The same reasoning as before, therefore, will suffice to determine the condition of equilibrium, which is, that the axis of the cylinder shall make with the horizon an angle not exceeding the angle of friction.

We are thus enabled in certain cases to solve a problem of

equilibrium by considering what movement or movements could take place if the equilibrium were broken, and then determining the conditions requisite in order that no one of these movements may be possible.

24. As another example, let the supporting surface consist of two inclined planes, whose line of intersection is horizontal, and whose slope is at the same side of the vertical, the upper plane being the steeper. Let a circular cylinder rest in a horizontal position on the two planes, being supported by a string coiled round its middle section, passing over a pulley and sustaining a weight, W'. To determine the conditions of equilibrium.

Here, if the equilibrium be broken, the motion may be any one of the three following:—1. The cylinder may descend the lower plane; 2. It may ascend the upper plane; 3. It may revolve round its axis without either ascending or descending.

Let the accompanying figure (Fig. 11) represent the middle section of the cylinder, IA, IB being sections of the supporting planes, P the pulley, W' the weight, PS the string, and VCV' a vertical. Then, if the cylinder be just on the point of descending the lower plane, there will evidently be no pressure on the upper plane, and the cylinder will be kept in equilibrium by three forces; namely, 1. The weight W' acting in the line $V'P$; 2. The weight of the cylinder itself W, acting in the line $V'C$; 3. The reaction of the plane IA, which must necessarily act in the line $V'T$. Hence, if CTV' be greater than the angle of friction for the plane IA, equilibrium is impossible. If CTV' be not greater than the angle of friction, the minor limit of the weight W' is evidently given by the equation,

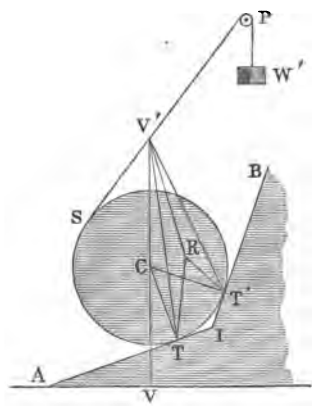

Fig. 11.

$$W' \sin TV'P = W \sin TV'C.$$

If the cylinder be on the point of ascending the upper plane, we must have, for a similar reason, $CT'V'$ not greater than the

angle of friction for IB, and the major limit of W' is then given by the equation,

$$W' \sin T'V'P = W \sin T'V'C.$$

If $CT'V'$ be greater than the angle of friction for IB, while CTV' is less than the angle of friction for IA; and if we suppose the weight W' to be gradually increased from its minor limit, the equilibrium will be broken by a rotation of the cylinder round its axis without either ascent or descent. To determine the value of W' for which this takes place, draw TR, $T'R$, making the angles CTR, $CT'R$ equal respectively to the angles of friction for the planes IA, IB. Then, if the cylinder be on the point of of slipping at T and T', the reactions of the two planes must act in the lines TR, $T'R$ respectively. The resultant of these reactions must, therefore, pass through R. But since the cylinder is kept in equilibrium by this force, and the weights W, W', acting respectively in the lines $V'C$, $V'P$, it is evident that the resultant of the reactions must pass through V', and must, therefore, act in the line $V'R$. In this case, therefore, the major limit of W' is given by the equation,

$$W' \sin RV'P = W \sin RV'C.$$

V. *Equilibrium of several Bodies.*

Prop. IV.

25. If a heavy body rest in a given position upon two others which themselves rest upon a horizontal plane, to determine the conditions of equilibrium of the system, and the friction actually developed at each point of contact, if it be determinate.

Consider, in the first place, the upper body. This is kept in equilibrium by three forces; namely, its own weight, W_2, and the two forces of resistance R, R' acting at the points of contact. These three forces must (as in Art. 15) be situated in the same vertical plane. Again, the lower bodies are kept in equilibrium respectively by the forces W_1, $-R$, R_1, and W_2, $-R'$, R_3. Each of these systems of three forces must, therefore, also be situated in a

vertical plane. But inasmuch as the systems W_2, R, R', and W_1, $-R$, R_1, are both situated in vertical planes and have a common line, namely, the line of direction of R and $-R$, the planes of these two systems must coincide. So also with the systems W_2, R, R', and $W_3 - R'$, R_3. Hence it is evident that the three centres of gravity and the four points of contact must be situated in the same vertical plane, which will also contain all the forces of the problem. Let this plane be represented by the plane of the paper, and let the accompanying figure (Fig. 12) represent the section of the system made by this plane.

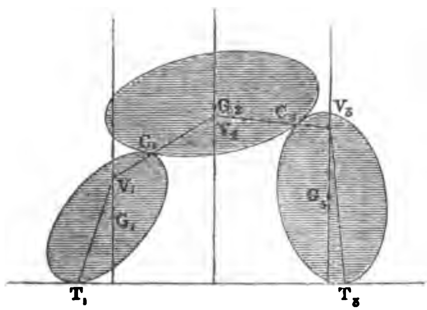

Fig. 12.

Let G_1, G_2, G_3, be the centres of gravity of the three bodies, and G_1V_1, G_2V_2, G_3V_3 vertical lines. Let T_1V_1 be the line of direction of R_1. Then it is evident that $V_1C_1V_2$ will be the line of direction of R, $V_3C_3V_2$ the line of direction of R', and T_3V_3 the line of direction of R_3. All these directions, therefore, are known if any one of them be known. Moreover, if the directions of these forces be known, the forces themselves are known. For since each of the bodies is kept in equilibrium by a system of three forces, we have

$$\left.\begin{aligned} R_1 &= W_1\frac{\sin G_1V_1C_1}{\sin T_1V_1C_1},\ R = W_1\frac{\sin G_1V_1T_1}{\sin T_1V_1C_1} = W_2\frac{\sin G_2V_2C_3}{\sin C_1V_2C_3}, \\ R_3 &= W_3\frac{\sin G_3V_3C_3}{\sin T_3V_3C_3},\ R' = W_3\frac{\sin G_3V_3T_3}{\sin T_3V_3C_3} = W_2\frac{\sin G_2V_2C_1}{\sin C_1V_2C_3}. \end{aligned}\right\}\quad (11)$$

These six equations involve five unknown quantities; namely, R_1, R_3, R_1, R'; and one of the angles which determine the directions of the forces. Eliminating these, we have one equation

among the given quantities of the problem, which, with the conditions respecting the centres of gravity and points of contact, form the only conditions of equilibrium which can be expressed by equations. Thus, for example, let the centres of gravity, G_1, G_2 be situated vertically over the respective points of contact. Then, inasmuch as two of the three forces which keep each of the lower bodies in equilibrium pass through the lower point of contact, T_1, the third force—namely, the pressure at the point C_1—must also pass through this point, and must, therefore, be directed in the line C_1T_1. In this case, therefore, the additional condition of equilibrium denotes that the lines T_1C_1, T_3C_3 intersect the vertical G_2V_2 in the same point. If the three bodies be spheres, this is necessarily true, as is easily seen.

The other conditions are, that no one of the directions found from the preceding equations for the forces of resistance shall make with the corresponding normal an angle greater than the angle of friction.

It is evident that if the vertical plane, which, as we have seen, contains all the forces, make with any one of the normals an angle greater than the angle of friction, equilibrium is impossible.

Prop. V.

26. A number of rough equal spheres are piled one upon another so as to form a pyramid, the lowest layer resting upon the ground. If there be no pressure between spheres belonging to the same horizontal layer, to determine whether the pressures at the several points of contact be determinate or indeterminate.

Let n be the number of spheres forming one side of the base of the pyramid. Then we know that if p be the total number of spheres contained in the base, and q the number of the remaining spheres contained in the pyramid—

$$p = \frac{n(n+1)}{1 \cdot 2}, \qquad q = \frac{n(n-1)(n+1)}{1 \cdot 2 \cdot 3},$$

$$p + q = \frac{n(n+1)(n+2)}{1 \cdot 2 \cdot 3}$$

The number m of distinct points of contact being three for each sphere not contained in the base, and one (with the ground) for each sphere contained in the base, is given by the equation,

$$m = 3q + p = \frac{n^2(n+1)}{1 \cdot 2}.$$

Now, at each point of contact there is a force, namely—the mutual action of the surfaces—which is unknown both in magnitude and in direction. Hence the number of unknown quantities introduced by these forces will be

$$3m = \frac{3n^2(n+1)}{1 \cdot 2}. \qquad (12)$$

But the number of equations of equilibrium, being six for each sphere, will be

$$n(n+1)(n+2).$$

Hence, if this number be less than $3m$, the forces of reaction are indeterminate. The limiting value of n for which the forces can be determinate, is found by equating these numbers. This gives $n = 4$. If n have this value, the equations of equilibrium will be in number equal to the unknown forces of reaction. But we cannot infer that these forces are, therefore, determinate. For it may be possible to eliminate all the forces of reaction from the equations of equilibrium, and to obtain thus a number of equations of condition, which, of course, give no assistance in determining these forces; or (as in the present case) one or more combinations of the equations of equilibrium may be identically true. It is, therefore, necessary to examine separately the cases $n = 2$, $n = 3$, $n = 4$.

For this purpose we shall somewhat generalize the question by supposing the bodies of which the pyramid is composed not to be spherical, the number in each layer being, however, the same.

1. Let $n = 2$. There are, in this case, three bodies resting upon the ground, each of which is kept in equilibrium by three forces; namely, gravity, the reaction of the ground, and the pressure of the single body which forms the apex of the pyra-

mid. These three forces must, therefore, be situated in the same plane. Hence it is easily seen that, for each of the lower bodies, the plane containing the centre of gravity and the two points of contact must be vertical. This condition, which is equivalent to three equations, is evidently identically true if the bodies be spheres. Again, since for each of the lower bodies the reactions of the ground and of the upper body are situated in a known plane, and have a known resultant, they are functions of a single unknown quantity. Hence the reactions which support the upper body are functions of three unknown quantities. Eliminating these unknown quantities from the six equations of equilibrium of the upper body, we have three additional equations of condition. In this case, therefore, the equations of equilibrium are sufficient to determine all the forces of resistance, and to furnish besides six equations of condition, which must be satisfied if the given position be one of equilibrium. This is in accordance with the general theory, in which we have seen that the number of equations of condition must be at least

$$n(n+1)(n+2) - \frac{3n^2(n+1)}{2} = 6 \text{ (if } n = 2).$$

If the bodies be equal spheres, the equations of condition are necessarily fulfilled. In fact, it is easy to see that in this case there is no condition of equilibrium which can be expressed by an equation. For it is evident, as in Prop. IV., that the pressure of the upper sphere on each of the lower spheres is directed in the line joining its point of contact with the upper sphere and its point of contact with the ground. It is evident, also, that the lines of direction of these pressures or of the corresponding reactions of the lower spheres intersect the vertical through the centre of the upper sphere in the same point. Considerations of symmetry show that the resultant of these reactions is directed in the vertical. Hence it is plain, that unless the line joining the two points of contact of any one of the lower spheres makes with the corresponding radius an angle greater than the angle of friction, it will always be possible to assign such values to the reactions of the ground and the spheres as will keep the whole system in equilibrium.

2. Let $n = 3$. In this case, according to the general theory, the equations of equilibrium are sufficient to determine all the forces of reaction, and to furnish besides,

$$\frac{3(3+1)}{2}(4-3)=6$$

equations of condition. The base of the pyramid consisting, in this case, of six bodies, each of the three bodies which are situated at the corner, is, as in (1), kept in equilibrium by two pressures and its own weight. Hence it is easy to see that for each of these bodies the plane containing the two points of contact and the centre of gravity must be vertical. This is, as before, the geometrical signification of three of the six equations of condition.

3. Let $n = 4$. In this case the formula (12) gives the number of equations of equilibrium equal to the number of the unknown quantities; namely, the intensities and directions of the reactions at the several points of contact. These unknown quantities are not, however, determinate, inasmuch as it is possible to form from the equations of equilibrium three equations not containing any of these unknown quantities. These equations express, as in the two previous cases, the geometrical condition that, for each of the bodies situated at the corners of the base, the centre of gravity and the two points of contact lie in the same vertical plane.

PROP. VI.

27. A number of solid bodies are connected by certain geometrical equations. To determine the geometrical forces corresponding to any one of these equations.

It is known that the position of a solid body in space depends upon six quantities, which may conveniently be assumed to be three co-ordinates fixing the position, in space, of some certain point in the body; and three angles, fixing the directions, in space, of any system of lines, which are fixed in the body. These angles may be taken in a variety of ways;

for convenience of physical interpretation, we shall suppose the angular position of the body to be fixed as follows:—

Conceive a system of rectangular lines, $x\ y\ z$, having a fixed direction in space, and a second system, $x'\ y'\ z'$, having the same origin as the first, and a fixed direction in the body. Let three planes be drawn through xx', yy', zz', respectively; and let ϕ be the angle between the first plane and xy, θ the angle between the second plane and yz, and ψ the angle between the third plane and zx. Then it is easily seen that these three angles fix the angular position of the body.

It is known that the most general movement which can be given to a solid body may be conceived to be made up of a movement of translation common to all its points, and a movement of rotation round any one point considered as fixed. Now, if the common origin of the axes $x\ y\ z$, $x'\ y'\ z'$ be taken to be the centre of rotation, $x\ y\ z$ being the co-ordinates of this point with regard to axes having a fixed origin and parallel to the first system of axes, it is plain that the components of a small motion of translation are δx, δy, δz, and the components of a small motion of rotation $\delta\phi$, $\delta\theta$, $\delta\psi$.*

Let $L = 0$ be one of the geometrical equations. Then it is plain that, in general,

$$L = \int (x, y, z, \phi, \theta, \psi, x_1, y_1, \&c.).$$

Then if X, Y, Z, X_1, &c., be the components of the resultant force for the several bodies, and M, N, P, M_1, &c., the components of the resultant movement; we have, by the principle of virtual velocities,

* It must be observed that $\delta\phi$, $\delta\theta$, $\delta\psi$ do not denote the actual rotations round the axes $x\ y\ z$ during the time δt. In the case of a motion of translation, a movement parallel to one axis causes no movement parallel to either of the other axes. But if a solid body be made to revolve round one axis, it will, in so doing, also revolve round each of the other axes. When we say, therefore, that $\delta\phi$, $\delta\theta$, $\delta\psi$ are the components of the motion of rotation, it is only meant that the complete rotation can be effected by causing the body to revolve successively round the three axes through these small angles. *Vid.* Routh's Rigid Dynamics, Chap. V.

$$X\delta x + Y\delta y + Z\delta z + X_1\delta x_1 + \&c. + M\delta\phi + N\delta\theta + P\delta\psi + M_1\delta\phi_1 + \&c.,$$

$$+ \lambda\left(\frac{dL}{dx}\delta x + \frac{dL}{dy}\delta y + \frac{dL}{dz}\delta z + \frac{dL}{d\phi}\delta\phi + \frac{dL}{d\theta}\delta\theta + \frac{dL}{d\psi}\delta\psi + \frac{dL}{dx_1}\delta x_1 + \&c.\right) + \&c. = 0;$$

λ being an indeterminate multiplier. Hence it is evident that the equation $L = 0$ may, so far as the body $(x, y, z, \phi, \theta, \psi)$ is concerned, be replaced by a system of forces whose resultant force is determined by the equations,

$$X' = \lambda\frac{dL}{dx}, \quad Y' = \lambda\frac{dL}{dy}, \quad Z' = \lambda\frac{dL}{dz};$$

and whose resultant moment is determined by the equations,

$$A = \lambda\frac{dL}{d\phi}, \quad B = \lambda\frac{dL}{d\theta}, \quad C = \lambda\frac{dL}{d\psi}.$$

It is evident that the replacing system will be equivalent to a single force if

$$\frac{dL}{dx}\frac{dL}{d\phi} + \frac{dL}{dy}\frac{dL}{d\theta} + \frac{dL}{dz}\frac{dL}{d\psi} = 0. \qquad (13)$$

This is most commonly the case in practice.

In general, we infer from the foregoing discussion that, as in the case of material particles, the forces which correspond to one geometrical equation are functions of a single unknown quantity λ.

PROP. VII.

28. A number of solid bodies connected by given geometrical equations rest upon fixed rough surfaces, and are acted on by given forces. To determine the forces of reaction at the several points of contact, so far as they are determinate.

Let n be the number of the solid bodies, and p, p_1, &c., the numbers of the supporting surfaces for each of these bodies respectively. Let also m be the number of the equations of condition. Then, as each point of contact introduces a force of

reaction unknown both in intensity and direction, the number of such unknown quantities will be $3(p + p_1 + \&c.)$.

Moreover, as we have seen in Prop. VI., each equation of condition, when replaced by the equivalent system of forces, introduces one unknown quantity. Hence the total number of unknown quantities is $3(p + p_1 + \&c.) + m$. But the number of equations of equilibrium, being six for each body, is altogether $6n$. If, then,

$$3(p + p_1 + \&c.) + m > 6n,$$

the equations of equilibrium are insufficient to determine the forces of resistance. Even if this condition be not fulfilled, these forces are not necessarily determinate. For it may be possible to eliminate all the unknown quantities from some group of the equations of equilibrium, and thus to obtain a number of equations of condition among the given quantities of the problem, necessary for equilibrium, but not giving any help towards the determination of the unknown forces of resistance. Thus, for example, let it be supposed that one of the solid bodies rests upon a single fixed surface, and that the number of equations of condition involving the position of this body does not exceed two. Then the six equations of equilibrium of this body cannot contain more than five unknown quantities; namely, the intensity and angles of direction of the single force of reaction, and the two (or one) quantities introduced by the geometrical equations. In this case, therefore, there must be at least one equation of condition among the given quantities of the problem, derived from the equations of equilibrium, and giving no help in the determination of the reactions of the other bodies.

This indeterminateness will necessarily occur if any one of the solid bodies rest upon more than one fixed surface. For suppose it to rest upon two. Then the six equations of equilibrium of this body will contain six unknown quantities; namely, the intensities and angles of direction of the two unknown reactions, which do not enter into any of the other equations. Now, it appears from Prop. III., that it is possible to eliminate these reactions from the equations of equilibrium, and thus to obtain a single equation denoting that the plane of the resultant mo-

ment of the remaining forces which act upon the body is parallel to the line joining the points of contact. Even then if the geometrical forces were all determined by the other equations, there would be but five equations to determine the forces of reaction at the two points of contact, thus leaving one unknown quantity undetermined.

Example 1.

29. Two cylinders act upon a rough inclined plane, and are connected by a string coiled round each of them at right angles to the axes which are horizontal. To find the position of equilibrium and the friction actually developed at the points of contact.

Assuming that the cord is coiled round the middle section of each cylinder, we may suppose the entire system to be reduced to these sections, and to be represented as in Fig. 13. Now, it is plain that the system of geometrical forces corresponding to the relation established between the two bodies by the cord is in this case equivalent to a single force; namely, the tension. This is in accordance with the general equation of condition (13). For, if we take the axes of x and y respectively parallel and perpendicular to AA', and the axis of z perpendicular to the plane of the paper, it is evident that the motion of translation will be parallel to x, and the motion of rotation round the axis of z. Hence, in this case,

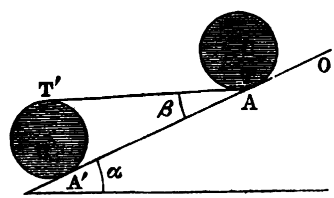

Fig. 13.

$$L = f(x, \psi, x', \psi');$$

whence it is plain that equation (13) is satisfied for both bodies.

Let T be the tension of the cord, W, W' the weights of the cylinders, and f, f' the effective forces of friction at A, A'. Then resolving parallel to AA', we have

$$W \sin a + T \cos \beta - f = 0, \quad W' \sin a - T \cos \beta - f' = 0. \quad (14)$$

Again, taking moments round the axes of the cylinders, we have

$$T = f = f';$$

whence

$$T(1 - \cos\beta) = W\sin\alpha, \quad T(1 + \cos\beta) = W'\sin\alpha.$$

Dividing these equations one by the other, we have

$$\tan^2 \tfrac{1}{2}\beta = \frac{W}{W'},$$

which determines the position of equilibrium. Again, substituting f or f' for T in the equations (14), we have

$$f = f' = \tfrac{1}{2}(W + W')\sin\alpha.$$

Let R, R' be the normal reactions at A, A'. Then resolving perpendicular to AA' we have

$$R + T\sin\beta - W\cos\alpha = 0, \quad R' - T\sin\beta - W'\cos\alpha = 0;$$

whence

$$R = W\cos\alpha - \sin\alpha\sqrt{WW'}, \quad R' = W'\cos\alpha + \sin\alpha\sqrt{WW'}.$$

The remaining condition of equilibrium is, therefore, that $(W + W')\sin\alpha$ shall not exceed the lesser of the two quantities,

$$2\mu(W\cos\alpha - \sin\alpha\sqrt{WW'}), \quad 2\mu(W'\cos\alpha + \sin\alpha\sqrt{WW'}.)$$

It is easy to verify that the limiting value of α obtained from this condition cannot exceed the angle of friction.

The greatest value which β can have, corresponding to the case in which the cylinders are in contact, is given by the equation,

$$\tan^2\frac{\beta}{2}\left(= \frac{W}{W'}\right) = \frac{r'}{r}.$$

Hence, if the cylinders be not in contact $Wr < W'r'$. The minor limit of the ratio of W to W' is derived from the equation $R = 0$, giving $W = W'\tan^2\alpha$. If the cylinders be in contact the question is modified by the friction which they mutually exert upon each other.

Example 2.

30. It is required to keep a carriage at rest upon an inclined plane by locking two of the wheels. To determine whether this can be done more effectually by locking the fore-wheels or the hind-wheels.

Assuming that the carriage is symmetrical with regard to its fore-wheels, and also with regard to the hind-wheels, we may, for the purposes of the present question, suppose the whole weight to be equally divided between two points, each of which is situated in the plane of one fore-wheel and one hind-wheel. It will then be sufficient to consider the equilibrium of one of the systems composed of these two wheels and the attached mass.

Let, then, the plane of the paper (Fig. 14) represent the plane of the two wheels, and G the point at which the weight is concentrated. Let TT' be the inclined plane, GN a perpendicular on TT', and GV a vertical.

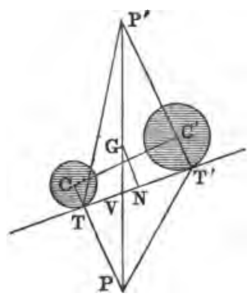

Fig. 14.

Then the system is kept in equilibrium by three forces; namely, the weight acting in the line GV, and the two reactions of the rough plane at the points T, T'. Now, it is plain that the reaction at the point of contact with the unlocked wheel is perpendicular to the inclined plane. Hence, if the fore-wheel be locked, the three forces act in the lines TP', $T'P'$, PV, and the system will slip as soon as the inclination of the plane is such that $CTP' > \epsilon$. For a similar reason, if the hind-wheel be locked, the system will slip when $CPT' > \epsilon$. If, therefore, $CTP' > CPT'$, it is better to lock the hind-wheels; and if $CTP' < CPT'$, it is better to lock the fore-wheels Now,

$$\tan CTP' = \frac{TT'}{T'N \cot \alpha + GN}, \quad \tan CPT' = \frac{TT'}{TN \cot \alpha - GN}.$$

It is, therefore, better to lock the fore-wheels, unless

$$TN - T'N > 2GN \tan \alpha,$$

indicating that the vertical through the centre of gravity cuts the inclined plane above the middle point of TT'.

The extreme inclination of the plane for which equilibrium is possible is found by putting

$$CTP' = \varepsilon, \text{ or } CPT' = \varepsilon,$$

according as the fore-wheels or hind-wheels are locked. In either case the inclination is less than the angle of friction. For it is evident from the figure that each of the angles CTP', CPT', is greater than a ($= TPP' = T'P'P$).

VI.—*Equilibrium of a Flexible String.*

Prop. VI.

31. A flexible string rests upon a rough surface, and is acted on by forces of any kind. To determine the conditions of equilibrium and the effective force of friction at each point, if it be determinate.

We shall find it convenient in the present case to consider the resistance of the rough surface as a single force compounded of the normal reaction and the effective force of friction (Chap. I., Art. 8). Let Xds, Yds, Zds be the rectangular components of this force acting upon an element of the string, ds. Let θ be the transverse section, and $\theta X_1 ds$, $\theta Y_1 ds$, $\theta Z_1 ds$, the components of the acting force. Let also T be the tension, and $x\ y\ z$ the co-ordinates of any point in the string. Then we have by the ordinary method,

$$\frac{d}{ds} T\frac{dx}{ds} + \theta X_1 + X = 0,$$

$$\frac{d}{ds} T\frac{dy}{ds} + \theta Y_1 + Y = 0, \qquad (15)$$

$$\frac{d}{ds} T\frac{dz}{ds} + \theta Z_1 + Z = 0.$$

Now, let it be supposed that the position of the string is given. Then, since $x\ y\ z$ are known functions of s, the foregoing equations contain the four unknown quantities, T, X, Y, Z. These

equations are, therefore, sufficient to determine any three of these quantities in terms of the fourth, which must remain indeterminate. This result is in strict accordance with that obtained for the case of a system of material particles (Art. 7). We saw there that, if the position of the system be given, the forces of effective resistance at the several points are fully determined by the equations of equilibrium in terms of the intensities of the geometrical forces which remain indeterminate.

Now, in the present case there is but one geometrical force; namely, the tension T. Hence, in accordance with the principle of Art. 7, the force of effective resistance at each point of the string should be determined in direction and magnitude in terms of T, which should remain indeterminate. We have seen that this is precisely the case.

If the string be at each point about to slip, the direction of effective resistance must be *on* the cone of resistance. We have thus an additional equation, and the problem becomes completely determinate. This is in accordance with the results obtained in Art. 8.

To express the condition that the direction of the force of effective resistance can never lie outside the cone of resistance. Let $\alpha\ \beta\ \gamma$ be the direction cosines of the normal to the supporting surface, and ϵ the angle of friction. Then we have, as in Art. 10:—

$$(\alpha X + \beta Y + \gamma Z)^2 > \cos^2\epsilon\ (X^2 + Y^2 + Z^2),$$

where the values of $X\ Y\ Z$ are to be substituted from equations 15. We shall now proceed to consider some particular cases of the general problem.

1. Let the string be acted on by forces at its extremities only. Let $a\ b\ c$ be the direction cosines of a perpendicular to the osculating plane to the curve. Then we know that

$$adx + bdy + cdz = 0, \quad ad^2x + bd^2y + cd^2z = 0.$$

Hence, if the equations (15) be multiplied respectively by a, b, c, and added, we shall have

$$aX + bY + cZ = 0.$$

The reaction of the rough surface, therefore, lies in the osculating plane to the curve formed by the string.

Now, let it be supposed that the string is about to slip at each point. Then, if the cone of resistance be described at any point of the curve, the direction of the force of reaction must lie upon this cone, and must, therefore, coincide with one of its intersections with the osculating plane. If, therefore, the curve of the string be given, the direction of the reaction is also given at each point as a function of ϵ. Let R be the intensity of this force, and l, m, n, its direction cosines. Then, if we eliminate R between any two of the equations (15); as, for example, between the first two, we shall have

$$\left(m\frac{dx}{ds} - l\frac{dy}{ds}\right)\frac{dT}{ds} + \left(m\frac{d^2x}{ds^2} - l\frac{d^2y}{ds^2}\right)T = 0.$$

But, since x, y, z, l, m, n, are known functions of s, this equation is of the form,

$$dT = T\phi(s)ds\text{; whence } T = T_0 e^{\int \phi(s)ds}\text{;}$$

T_0 being the value of the force of tension at one extremity of the string, which is evidently equal to the applied force at this point. This force must evidently be tangential. If T_1 be the applied force at the other extremity, we must have

$$T_1 = T_0 e^{\int_0^s \phi(s)ds}.$$

This is the relation which must exist between the forces which act at the extremities, if the string be about to slip.

The geometrical meaning of $\phi(s)$ is readily found by taking the tangent to the curve as the axis of x, and the radius of absolute curvature as the axis of y. We have then

$$\frac{dx}{ds} = 1, \quad \frac{dy}{ds} = 0, \quad \frac{d^2x}{ds^2} = 0, \quad \frac{d^2y}{ds^2} = -\frac{1}{\rho},$$

ρ being the radius of curvature. Moreover, since the force of reaction lies in the plane of xy, if θ be the angle which the direction of this force makes with the radius of curvature, we have

$$\cos\theta = m, \quad \sin\theta = \mp l\text{;}$$

the ambiguity of sign resulting from the fact that the force of reaction may be directed in either of the two lines of intersection of the osculating plane with the cone of resistance.

Assuming the tangent to be measured in the same direction with s, it is easily seen that one of these lines makes with it an acute angle, and the other an obtuse angle; the upper sign in the value of sin θ corresponding to the former of these cases, and the lower sign to the latter. Substituting these values in the value of $\phi(s)$, we have

$$\phi(s) = \mp \frac{\tan \theta}{\rho}.$$

Hence, putting $ds = \rho d\omega$, the value of T becomes

$$T = T_0 e^{\mp \int \tan \theta d\omega};$$

showing that T augments or diminishes according as the direction of the force of reaction makes an obtuse or an acute angle with the direction of s. This is evident in itself.

If the string be about to slip at each point in the direction of its length, the effective force of friction is directed in the tangent to the curve. Since, therefore, the complete reaction, R, of the surface is compounded of the normal reaction and the effective force of friction, it is evident that the plane passing through R and the tangent to the curve, which is the osculating plane, is normal to the surface. The curve, therefore, must be a geodetic line.

It is also true, conversely, that if the string form a geodetic curve, the friction at each point is necessarily directed in the tangent.

Moreover, since the radius of curvature coincides in direction with the normal to the surface, θ will denote the angle which R makes with the normal; and since the string is about to slip, this angle must be at every point equal to the angle of friction. We have then,

$$\tan \theta = \mu, \text{ and } T = T_0 e^{\mp\mu(\omega - \omega_0)}.$$

A familiar instance of the general theorem is found in the case of two weights attached to the ends of a thin string coiled

round a rough horizonal cylinder at right angles to the axis. Neglecting the thickness of the string, we may suppose it to be coiled n times round the cylinder. We shall have then,

$$\omega_1 - \omega_0 = (2n + 1)\pi, \text{ and } T_1 = T_0 e^{\pm(2n+1)\mu\pi}.$$

Hence, if W_0, W_1 be the suspended weights, the extreme ratios of W_1 to W_0 consistent with equilibrium are given by the equations,

$$W_1 = W_0 e^{(2n+1)\mu\pi} \quad W_1 = W_0 e^{-(2n+1)\mu\pi}.$$

These results are true for a cylinder of any form if it have no singular lines.

2. As another example, we shall consider the case of a heavy chain wrapped round a rough horizontal cylinder. Let the axis of x be parallel to the axis of the cylinder, the axis of z vertical, and the axis of y perpendicular to the other two. Let the chain be wrapped round the cylinder in the form of a helix, each spire being in contact with the two adjacent spires. Let a be the radius of the cylinder, and b the width of the chain. Then the equations of either edge of the chain will be

$$y^2 + z^2 = a^2, \quad z = y \tan \frac{2\pi x}{b};$$

or, assuming $\qquad 2\pi x = b\omega$ and $2\pi m = b$,

$$y = a\cos\omega, \quad z = a\sin\omega, \quad x = m\omega, \quad ds = d\omega\sqrt{a^2 + m^2}.$$

Now, if we conceive a section of the chain bounded by two curves parallel to either edge, it is evident that the equations of equilibrium of this section will be the same as those of any other similar section, and may therefore be taken for the equations of equilibrium of the chain itself. Introducing then the foregoing values into the equations of equilibrium (15), and putting $n^2 = a^2 + m^2$, we have (since $X_1 = 0$, $Y_1 = 0$, $Z_1 = -g$),

$$m\frac{dT}{d\omega} + n^2 X = 0, \; a\left(\sin\omega\frac{dT}{d\omega} + T\cos\omega\right) - n^2 Y = 0, \tag{16}$$

$$a\left(\cos\omega\frac{dT}{d\omega} - T\sin\omega\right) + n^2(\theta g - Z) = 0.$$

If the chain be about to slip, we have the further condition that the reaction of the cylinder must lie upon the cone of resistance. This condition gives the equation,

$$X^2 + Y^2 + Z^2 = \sec^2\epsilon\,(Z\sin\omega - Y\cos\omega)^2, \qquad (17)$$

in which the values of X, Y, Z, are to be substituted from the equations (16).

These equations admit of integration, if we neglect the width of the chain in comparison with the radius of the cylinder, in which case the curve of the chain becomes $q\,p$, a circle. We have then $m = 0$, $n = a$. Hence, from the first equation (16) $X = 0$. This condition enables us to reduce the equation (17) to the form

$$(Y\sin\omega + Z\cos\omega)^2 = \tan^2\epsilon\,(Y\cos\omega - Z\sin\omega)^2;$$

whence

$$Y\sin\omega + Z\cos\omega \pm \tan\epsilon\,(Y\cos\omega - Z\sin\omega) = 0;$$

substituting for Y and Z from the equations (16), we have

$$\frac{dT}{d\omega} + \theta ga\cos\omega \pm \tan\epsilon\,(T - \theta ga\sin\omega) =$$

the upper or lower sign being taken according as the slipping is *towards* or *from* the origin. Assuming that the slipping is about to take place *from* the origin, and integrating the equation upon this hypothesis, we have

$$T = ce^{\omega\tan\epsilon} - \theta ga\sin(2\epsilon - \omega),$$

c being an arbitrary constant. To determine c, let l be the length of one of the vertical parts of the chain. We have then at the first point of contact with the cylinder,

$$T = \theta gl, \quad \omega = 0;$$

whence

$$c = \theta g\,(l + a\sin 2\epsilon).$$

Let l' be the length of the other vertical part of the chain. Then the greatest value of l' consistent with equilibrium is found by the equation

$$l' = (l + a\sin 2\epsilon)\,e^{(2n+1)\pi\tan\epsilon} + a\sin 2\epsilon,$$

the chain being supposed to be wrapped n times round the cylinder. If l and l' be increased indefinitely, this result agrees, as it ought, with that in page 56 for the cord.

CHAPTER III.

ON EXTREME POSITIONS OF EQUILIBRIUM.

1. A MATERIAL particle may be said to be in an extreme position of equilibrium when any diminution in the coefficient of friction of the surface on which it rests would cause the equilibrium of the system to be broken. Thus, for example, in the case of an isolated particle, the position of equilibrium is extreme, if the acting force be directed along a side of the cone of resistance. For, in this case, if the coefficient of friction were diminished by any quantity, no matter how small, the cone of resistance would be contracted, the acting force would fall outside, and therefore (p. 12) the equilibrium would no longer exist.

Again, a finite body which rests on more than one surface may be said to be in an extreme position of equilibrium, if any one of the points of contact be in an extreme position; that is to say, if the equilibrium would be broken by the diminution of the coefficient of friction of the supporting surface at that point. An important class of questions arises with regard to these positions. When a number of rough bodies are in contact with each other, it is not in general necessary to the rupture of equilibrium that slipping should take place at *every* point of contact. It frequently happens that, so far as the geometric conditions of the system are concerned, many different kinds of motion are equally possible. Thus, for example, if a double ladder be placed in an upright position on a horizontal plane, and if its two sides (supposed to be of unequal length) be gradually drawn out from each other till equilibrium be broken, the initial motion may be one of three kinds; namely, slipping at either end accompanied by rotation round the other end, or slipping at both ends simultaneously. Similarly, if a beam be placed with one extremity on the ground and the other resting against a wall, and if its position be gradually changed till equilibrium is broken, the initial motion may either be slipping at both extremities simultaneously, or slipping at the upper extremity combined with rotation round the lower. If the beam be so placed that

an indefinitely small diminution of the coefficient of friction at the extremities would cause motion of the second kind, and could not cause motion of the first kind, the upper end alone is said to be in an extreme position. If the beam be so placed that such a diminution might* cause motion of the second kind, both ends are said to be in extreme positions.

2. Instead of an indefinitely small diminution of the coefficient of friction we may substitute an indefinitely small change in the force, external or geometrical, which acts upon the particle; this change being so made as to render the resultant *more* oblique to the normal. For the effect of both kinds of change is the same—namely, to throw the line of direction of the acting force *without* the cone of resistance, and therefore to cause the equilibrium to be broken. Such a change may be produced by an indefinitely small shock, or instantaneous force communicated to the particle. If, therefore, we use the word "disturbance" to denote such a force, we may say that the position of a particle is extreme if an indefinitely small disturbance would cause it to slip. But in thus enunciating the definition, it must be carefully observed that "disturbance" is quite distinct from "displacement" or "velocity." For, as we shall hereafter see, there are cases in which an indefinitely small *velocity* communicated to a particle *not* in an extreme position, as here defined, may cause the particle to pass into a state of finite motion.

It becomes then an important question, in the case of a system which is in an extreme position of equilibrium, to decide which of the several geometrically possible movements may be expected actually to happen if the system experience a very small disturbance. We have to determine at which of the points of contact slipping may take place; or, in the terms above explained, which of these points are in extreme positions of equilibrium. Now, one characteristic of an extreme position of

* I say "might cause;" for even though both ends be in extreme positions, it does not follow that both extremities will actually begin to move. But if both ends be in extreme positions, an indefinitely small disturbance in the system *may* cause motion at both ends. Whether the actual initial motion be at one or both ends will depend upon the nature of the disturbance which has been communicated to the system, and must be determined dynamically. We shall return to this subject in a subsequent chapter.

equilibrium is this, that the effective resistance of the surface at that point shall be directed along a side of the cone of resistance. For, unless the line of direction of the effective resistance were so placed, a small diminution in the coefficient of friction would not throw this line without the cone, and therefore (p. 13) the equilibrium would still be preserved.

If the system be of such a nature that in its positions of equilibrium the effective forces of resistance are determinate, at least in direction, this characteristic is sufficient, and there is no theoretic difficulty in determining whether any given position of equilibrium be an extreme position, and also at what point or points the motion of slipping may be expected to happen, if the equilibrium be broken in the manner above described. For we have only to determine the angles of effective resistance at the several points of contact in functions of the given quantities of the problem. Then, if we find that at any one or more of these points these angles have their maximum values, we conclude that the points at which such values occur are in extreme positions, and that a motion of slipping may be expected to occur at any or all of these points, if the system receive a small disturbance.

If the directions of the effective forces of resistance at the several points of contact be not determinate, the problem is one of somewhat greater difficulty. In all cases, however, whether these directions be determinate or indeterminate, the two following principles are sufficient for its solution:—

1. If any point of contact be in an extreme position of equilibrium, the angle of effective resistance at that point must have its extreme value $\tan^{-1}\mu$.

2. Any value less than this at that point must be inconsistent with some of the conditions of equilibrium.

Hence we infer for the case where the directions of effective resistance are indeterminate,—

1. That, at every point of contact which is in an extreme position, the value $\tan^{-1}\mu$ must be an *admissible* value for the angle of effective resistance.

2. That at that point this value must be the *least* which will satisfy the condition of equilibrium.

If, then, a system contain a number of points of contact,

some of which are, and some of which are not, in extreme positions, the relations of such a system must be satisfied by the following system of values assigned to the angles of effective resistance:—1. For those points which are in extreme positions the values $\tan^{-1}\mu_1$, $\tan^{-1}\mu_2$, &c. 2. For the points which are not in extreme positions, a system of values, each of which differs by a finite quantity from the corresponding extreme value. Moreover, it must be impossible, for the first class of points, to diminish any one of the values so assigned to the angles of effective resistance at these points, without increasing some one or more of the others. For, as we have seen, any such diminution must be inconsistent with some one or more of the conditions of equilibrium. It must, therefore, be inconsistent either with some one of the geometric relations, in which case the value in question is evidently an absolute minimum, or with the condition that no one of these angles shall exceed the corresponding extreme values $\tan^{-1}\mu_1$, $\tan^{-1}\mu_2$, &c. If, therefore, the value in question be not an absolute minimum, it must, at least, be impossible to diminish it without at the same time increasing the angle of effective resistance at some other point which is also in an extreme position. If one point only of the system be in an extreme position, the value of the angle of effective resistance at that point must evidently be an absolute minimum.

3. Before entering upon the general problem, we shall consider some particular cases which admit of easy geometrical solutions:—

1. Let two beams, connected at their upper ends by a smooth hinge, be placed with their lower extremities resting on a rough horizontal plane, and let it be required to determine whether the equilibrium be on the point of being broken; and, if so, in what way this will happen.

Let AC, BC (Fig. 15) be the beams; their plane, which is necessarily vertical, being that of the paper.

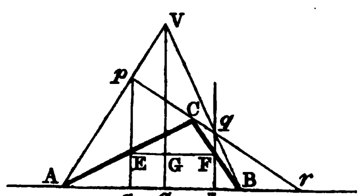

Fig. 15.

Then the beam AC is kept in equilibrio by—1. The weight acting in the vertical which passes through its centre of gravity, E. 2. The reaction of the hinge acting at A. 3. The reaction of the hinge acting at C.

The direction of these forces must, therefore, pass through the same point. Let pE, Ap, Cp be the lines of direction of these forces, and qF, Bq, Cq the corresponding directions for the second beam. Then, since the hinge is smooth, the lines Cp, Cq, lie *in directum*. Now, considering the two beams as forming a single system, this system is kept in equilibrium by—1. The weight of the system acting in a vertical through the centre of gravity, G. 2. The reaction of the rough plane acting in the line Ap. 3. The reaction of the rough plane acting in the line Bq. These forces must pass through the same point, V. The determination of the reactions at the points A, C, B, is therefore reduced to the following geometrical question:—To draw a line through C intersecting the verticals pE, qF in p and q, so that the lines Ap, Bq may meet on the vertical gG. This is evidently done by making

$$\frac{Ar}{Br}\left(=\frac{Ap,\ Vq}{Vp,\ Bq}\right)=\frac{Aa\times gb}{Bb\times ga}.$$

The directions of the reactions at A, B being thus ascertained, if either of these directions make with the corresponding normal an angle equal to the limiting angle for that plane, the system is about to slip at that point; and if the angle exceed the limiting angle, the position is not one of equilibrium.

Thus, for example, let the beams be uniform, and of the same thickness, and resting upon a rough horizontal plane. Then it is easy to see that if $Ac > Bc$, then $Ag > Bg$; and, therefare, $AVg > BVg$. Hence the system will slip at the extremity of the longer beam.

2. As another instance, let us take the case of a beam, AB, $B'A'$ (Fig. 16), resting with one extremity on the ground, to which it is attached by a smooth hinge, and supported by a rough circular cylinder (also resting on the ground), to whose axis the beam is at right angles. Let the coefficients of friction be the same for the beam and the ground, and let it be required to determine whether either of the points G T be in an extreme position of equilibrium.

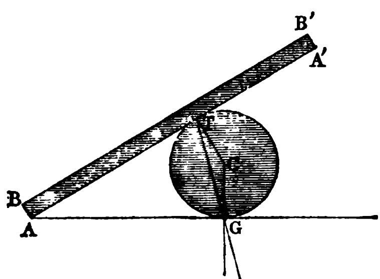

Fig. 16.

It is evidently sufficient to consider sections of the cylinder and beam made by a plane perpendicular to the axis of the former. These are represented in Fig. 16. We may then consider the cylinder to be kept in equilibrium by three forces, namely—1. The weight directed in *CG*. 2. The reaction of the ground acting at *G*. 3. The reaction of the beam acting at *T*. As these forces must, for equilibrium, pass through the same point, it is evident that the reaction of the beam must pass through *G*, and must therefore be directed in the line *TG*. Hence it is evident that the reaction of the ground must lie somewhere *within* the angle *TGC*, and must, therefore, make with the normal *CG* an angle *less* than *TGC*, and therefore less than *GTC*, the angle which the reaction of the beam makes with the normal *CT*. Hence *G* cannot be in an extreme position. The other point, *T*, will be in an extreme position if *GTC* be equal to the angle of friction. This conclusion may be otherwise stated by saying that, if the cylinder be slowly pushed towards *A*, equilibrium will be broken by slipping at *T*, which will happen when *GAT* = double the angle of friction.

3. As another instance, let us take the case of a beam resting against a rough cylinder, and on a rough horizontal plane, upon which the cylinder also rests, its axis being at right angles to the beam; and let it be required to determine whether or not the equilibrium is on the point of being broken; and, if so, in what way this will happen.

Let the plane of the paper (Fig. 17) represent a section of the beam and cylinder at right angles to the axis of the latter, and passing through the centre of gravity of the beam. Then, so far as the geometrical relations of the system are concerned, the equilibrium may be broken by slipping at any of the points *A*, *T*, *R*. It is evident, as in the foregoing example, that slipping must take place at *R* before it takes place at *T*. It remains to consider whether it will take place first at *R* or at *A*. Now, the obliquity of the reaction at *R* = *CRT* = *CTR* = *TVG*. And the obliquity of the reaction at

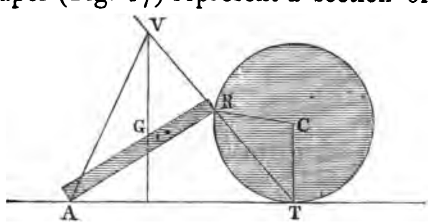

Fig. 17.

$A = AVG$. But it is easily seen that, if the beam be uniform, $TVG > AVG$. Hence the rupture of equilibrium will be by slipping at R, rolling at T, and rotation of the beam round A.

PROP. I.

4. A number of material particles are situated each on a rough surface, connected with each other by known relations, and acted on by given forces. To determine whether any given number of these particles be in extreme positions of equilibrium.

Let n be the number of the material particles, and let them be denominated 1, 2, 3, ... n. Let the particles 1, 2, 3, ... k be in extreme positions of equilibrium; that is to say, in such positions that, if the coefficient of friction of the surface on which any one of these particles rests were diminished, the equilibrium would necessarily* be broken; the position of these particles must then satisfy the following conditions:—

1. It must be possible, consistently with the conditions of the system, to assign to the angles of effective resistance at the points 1, 2, ... k, their extreme values.

2. It must not be possible, consistently with these conditions, to diminish the value of any one of these angles without increasing that of one or more of the others. We shall now proceed to consider how these conditions are mathematically expressed.

Let m be the number of the equations of condition, and R_1, R_2 . . . . R_m the geometrical forces resulting from these equations. Let A_1, A_2 . . . . A_n be the resultants of the given forces at the points 1, 2, 3, . . . n respectively, and let us denote by the symbols

$$\overline{A_1 R_1}, \; . \; . \; . \; . \; \overline{R_1 R_2} \; . \; . \; . \; .$$

the angles between the directions of A_1 and R_1 . . . . . and of R_1 and R_2 . . . . Then if u_1, u_2 . . . . u_n be the cosines of the angles which the directions of effective resistance at the several

* The word "necessarily" is important. For, the position of the system being given, the values of the geometrical forces may be such that rupture of equilibrium would follow a diminution in the coefficient of friction at any one of the points 1, 2, . . . k, while at the same time these forces are capable of values which would permit such a diminution without any rupture of equilibrium. This point will be more fully considered when we come to treat of "Necessary and possible equilibrium."

points make with the normals to the respective surfaces, we have, denoting these normals by $N_1, N_2 \ldots . Nn$,

$$u_1 = \frac{\text{normal component of total force}}{\text{total force,}}$$

$$= \frac{A_1 \cos \overline{A_1 N_1} + R_1 \cos \overline{R_1 N_1} + R_2 \cos \overline{R_2 N_1} + \&c.,}{\sqrt{A_1^2 + R_1^2 + R_2^2 + \&c. + 2A_1R_1 \cos \overline{A_1R_1} + \&c. + 2R_1R_2 \cos \overline{R_1R_2} + \&c.}}$$

the number of the geometrical forces R_1 R_2, &c., which enter into the value of u_1, being the same as the number of the equations of condition which involve the co-ordinates of the point 1. Similar values are, of course, found for u_2, u_3, &c. Each of these cosines is, therefore, a function of a certain number of the quantities R_1, R_2, &c., which are, so far at least as conditions of equilibrium are concerned, indeterminate. We may represent them generally by the equations

$$u_1 = \phi_1 (R_1, R_2 \ldots .), \quad u_2 = \phi_2 (R_1 \ldots .), \text{ \&c.}$$

Denoting then by a_1, a_2 . . . . . the cosines of the angles of friction, the first of the above-mentioned conditions requires that the equations

$$u_1 = a_1, \; u_2 = a_2, \ldots . \; u_k = a_k,$$

should be satisfied by real values of R_1, R_2, &c.

Again, differentiating the values of u_1, u_2, &c., with respect to R_1, R_2, &c., &c., we have

$$du_1 = \frac{du_1}{dR_1} dR_1 + \frac{du_1}{dR_2} dR_2 + \&c.,$$

$$du_2 = \frac{du_2}{dR_1} dR_1 + \frac{du_2}{dR_2} dR_2 + \&c.,$$

&c., &c.,

$$du_k = \&c.$$

The second of the conditions requires that the cosines u_1, u_2 . . . u_k should be incapable of being all simultaneously increased.

K

Hence it must be impossible to assign to dR_1, dR_2 . . . . such values as will make all the increments

$$du_1,\ du_2,\ \ldots\ \&c.\ du_k,$$

positive.

Before proceeding to consider how these conditions are to be fulfilled, it will be necessary to premise the following Lemmas:—

PROP. II.

Let $x_1, x_2, \ldots x_m$ be a series of variables limited by one only restriction, namely, that no one of them shall be negative.

Let $u_1, u_2, u_3 \ldots u_n$ be a series of functions of the form

$$u_1 = A_{1,0} + A_{1,1}\,x_1 + A_{1,2}\,x_2 + \&c. + A_{1,m}\,x_m,$$

$$u_2 = A_{2,0} + A_{2,1}\,x_1 + A_{2,2}\,x_2 + \&c. + A_{2,m}\,x_m,\ \&c., \&c.,$$

$$u_n = A_{n,0} + A_{n,1}\,x_1 + A_{n,2}\,x_2 + \&c. + A_{n,m}\,x_m,$$

where $n > m$.

To determine the conditions requisite in order that it may be possible to assign such values to $x_1\ x_2, \ldots x_m$ as will make $u_1, u_2, \ldots u_n$ all positive.

1. It is plain that no series of coefficients such as $A_{1,0}, A_{1,1}, \ldots A_{1,m}$ in the same horizontal line can be all negative. For if this were the case, no system of positive values assigned to $x_1, x_2 \ldots x_m$ would render u_1 positive.

2. If any such series of coefficients be positive, it is unnecessary to take any account of the function u_1. For inasmuch as *any* system of positive values assigned to x_1, x_2, &c., would make u_1 positive, the required condition must be independent of this function.

3. If any vertical series, as $A_{1,1}, A_{2,1}$, &c., be all positive, it is always possible to make all the functions positive by taking the corresponding variable x_1 sufficiently large.

4. If any such vertical series be all negative, we may always take $x_1 = 0$, and this will obviously be the most favourable value.

Hence in the first and third cases we are enabled to give an

immediate answer to the question, while in the second and fourth the subsequent operations may be simplified by leaving out of consideration either one of the functions or one of the variables.

Let us suppose that in the first p of the functions u_1, u_2, &c., the coefficient of x_1 is positive, and in the remaining functions negative. Then the first p functions will be positive if

$$A_{1,1}x_1 > -(A_{1,0} + A_{1,2}x_2 + \&c.),$$

$$A_{2,1}x_1 > -(A_{2,0} + A_{2,2}x_2 + \&c.),$$

$$\&c., \quad \&c.;$$

$$A_{p,1}x_1 > -(A_{p,0} + A_{p,2}x_2 + \&c.).$$

In the remaining $n-p$ functions we must have

$$-A_{p+1,1}x_1 < A_{p+1,0} + A_{p+1,2}x_2 + \&c.$$

$$-A_{p+2,1}x_1 < A_{p+2,0} + A_{p+2,2}x_2 + \&c.$$

$$\&c., \quad \&c.;$$

$$-A_{n,1}x_1 < A_{n,0} + A_{n,2}x_2 + \&c.$$

We have thus p inequalities of the form

$$x_1 > b_1, x_1, > b_2 \; . \; . \; . \; . \; x_1 > b_p,$$

and $n-p$ inequalities of the form

$$x_1 < b_{p+1}, \;\; x_1 < b_{p+2} \; . \; . \; . \; . \; x_1 < b_n$$

From these we deduce immediately the conditions

$$b_{p+1} > b_1, \;\; b_{p+1} > b_2, \; . \; . \; . \; . \; b_{p+1} > b_p,$$

$$b_{p+2} > b_1, \; . \; . \; . \; . \; . \; .$$

$$. \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; .$$

$$b_n > b_1 \; . \; . \; . \; . \; . \; . \; . \; . \; b_n > b_p.$$

We must have besides the further conditions,

$$b_{p+1} > 0, \quad b_{p+2} > 0 \; . \; . \; . \; . \; b_n > 0.$$

The remaining variables $x_2, x_3, \ldots x_m$ must, therefore, be such as to make the functions

$$b_{p+1} - b_1, \quad b_{p+1} - b_2, \quad \ldots \ldots \quad b_{p+1} - b_p,$$

$$b_{p+2} - b_1, \quad b_{p+2} - b_2, \quad \ldots \ldots \quad b_{p+2} - b_p,$$

$$\&c., \qquad \&c., \qquad \&c.;$$

$$b_n - b_1, \quad b_n - b_2, \quad \ldots \ldots \ldots \quad b_n - b_p,$$

$$b_{p+1}, \quad b_{p+2}, \quad \ldots \ldots \ldots \ldots \quad b_n;$$

all positive.

Arranging these functions, which are evidently $(p+1)(n-p)$ in number, and contain only $m-1$ variables, we have $(p+1)(n-p)$ conditions of the form

$$B_{1\,0,} + B_{1,2}\,x_2 + \&c. + B_{1,m}\,x_m > 0,$$

$$B_{2\,0,} + B_{2,2}\,x_2 + \&c. + B_{2,m} x_m > 0,$$

$$\&c., \qquad \&c, \qquad \&c.$$

These functions are to be treated in the same way as u_1, u_2, &c.

Thus, if any horizontal series of coefficients be all negative, we infer that the problem is impossible; and if any vertical series be all positive, we infer that the problem is always possible. Also, as before, if any horizontal series be all positive, or any vertical series all negative, the corresponding function or variable may be neglected.

If we do not find either a negative horizontal series or a positive vertical series, the same process must be continued until such a series be arrived at, or until all the variables be eliminated. In the latter case it is plain that we shall have a number of conditions of the form

$$L > 0, \quad M > 0, \quad \&c.$$

L, M, &c., being functions of the coefficients alone, without any of the variables, in the series for u_1, u_2, &c.

Prop. III.

To determine the conditions requisite in order that it may be possible to assign to $x_1, x_2, \ldots\ldots x_m$ *any* real system of values, positive or negative, which will render $u_1\ u_2, \ldots u_n$ all positive.

If n be less than or equal to m, it will in general be possible

to find an infinite number of such systems. For it is only necessary to equate $u_1, u_2, \ldots u_n$, respectively, to any series of positive quantities, $a_1, a_2, \ldots a_n$, and from these equations to determine $x_1, x_2, \ldots x_m$. (If n be less than m, we may add $m-n$ equations *ad lib.*, so as to make up the number necessary for the determination of $x_1\ x_2, \ldots x_m$). But it may so happen that in the elimination of some of the quantities $x_1, x_2, \ldots$ the rest may disappear with them, leaving one or more equations of the form

$$L_1a_1 + L_2a_2 + \&c. + L_na_n = 0,$$

or

$$L_1u_1 + L_2u_2 + \&c. + L_nu_n = 0. \qquad (1)$$

Let the number of such equations be p, and let them be solved for $u_1\ u_2, \ldots u_p$; we shall have then results of the form

$$u_1 = M_{p+1}u_{p+1} + M_{p+2}u_{p+2} + \&c.,$$
$$u_2 = \&c.$$
$$\&c., \qquad \&c.$$

It must, therefore, be possible to assign such positive values to $u_{p+1}, u_{p+2}, \ldots u_n$ as will render the functions

$$M_{p+1}u_{p+1} + M_{p+2}u_{p+2} + \&c.$$
$$\&c. \qquad \&c.,$$

positive. This case is reduced, therefore, to Prop. II.

If n be greater than m, the elimination of x_1, x_2, &c., can, of course, be always effected, giving one or more equations of the form (1). The remainder of this case is as before.

Prop. IV.

Let $u_1, u_2, \ldots u_m$ be any functions of the n variables $x_1, x_2, \ldots x_n$, and let $a_1, a_2, \ldots a_m$ be a given system of values of these functions. To determine the conditions requisite in order that it may be impossible to raise any of the functions above the given values without diminishing one or more of the others below their corresponding values.

If this augmentation be impossible, it is plain that some one or more of the differentials

$$du_1,\ du_2,\ \ldots\ldots\ du_n,$$

or

$$\frac{du_1}{dx_1}dx_1 + \frac{du_1}{dx_2}dx_2 + \&c. + \frac{du_x}{dx_m}dx_m,$$

$$\frac{du_2}{dx_1}dx_1 + \frac{du_2}{dx_2}dx_2 + \&c. + \frac{du_2}{dx_m}dx_m,$$

$$\&c., \qquad \&c., \qquad \&c.,$$

must be negative when $u_1, u_2 \ldots\ldots$ are replaced by $a_1, a_2 \ldots\ldots$

It must, therefore, be impossible to assign such values to dx_1, dx_2, &c., as will make any of the increments du_1, du_2, &c., positive without making some one or more of the rest negative. The problem is, therefore, similar to that discussed in Prop. III., putting du_1, du_2, &c., in place of u_1, u_2, &c., and

$$\frac{du_1}{dx_1},\ \frac{du_1}{dx_2},\ \&c.,$$

$$\frac{du_2}{dx_1},\ \&c.,$$

in place of $A_{1,1}\ A_{1,2}$, &c.

If, therefore, it be impossible to fulfil the conditions there spoken of, it will be impossible to augment all the quantities u_1, u_2, &c., beyond the given series of values.

We infer, therefore, that if the augmentation be impossible, there must exist one or more equations of the form

$$L_1du_1 + L_2du_2 + \&c. = 0, \qquad (2)$$

and that it must be impossible to satisfy these equations by any system of positive values of the quantities du_1, du_2, &c.

If the number of the functions u_1, u_2, &c., be greater than that of the variables x_1, x_2, &c. one or more equations, such as (2), can always be found by eliminating dx_1, dx_2, &c. from the equations

$$du_1 = \frac{du_1}{dx_1}\,dx_1 + \frac{du_1}{dx_2}\,dx_2 + \&c.$$

$$du_2 = \frac{du_2}{dx_1}\,dx_1 + \&c. \qquad (3)$$

$$\&c. \qquad \&c.$$

If the number of the functions u_1, &c. be less than that of x_1, &c., the conditions requisite for the existence of one or more equations of the form (2) will be found by multiplying the equations (3) respectively by the indeterminate factors λ_1, λ_2, &c., adding and equating to zero the coefficient of each of the increments dx_1, dx_2, &c.

Eliminating then the $n - 1$ ratios,

$$\frac{\lambda_2}{\lambda_1}, \frac{\lambda_3}{\lambda_1} \cdots\cdots\cdots \frac{\lambda_n}{\lambda_1},$$

we have $m - n + 1$ equations of condition among the quantities

$$\frac{du_1}{dx_1}, \frac{du_1}{dx_2}, \&c.,$$

$$\frac{du_2}{dx_1}, \&c.,$$

$$\&c., \&c.$$

These equations must hold for all values of x_1, x_2, &c., consistent with the conditions

$$u_1 = a_1, \qquad u_2 = a_2, \&c.$$

If this condition be fulfilled, there will be one equation of the form

$$L_1 du_1 + L_2 du_2 + \&c. = 0.$$

If there be two or more equations of this form, the existence of such equations will appear in the process of eliminating the ratios,

$$\frac{\lambda_2}{\lambda_1}, \frac{\lambda_3}{\lambda_1}, \&c.,$$

in the same way in which the first appeared in the elimination

of x_1, x_2, &c. Thus, for example, if the elimination of $p-1$ of these ratios among the first p of the equations,

$$\begin{aligned}
\lambda_1 \frac{du_1}{dx_1} + \lambda_2 \frac{du_2}{dx_1} + \&c. &= 0, \\
\lambda_1 \frac{du_1}{dx_2} + \&c. \qquad\quad &= 0, \\
\&c. \qquad \&c.,&
\end{aligned} \tag{4}$$

cause all the rest to disappear, we shall have only $p-1$ equations between the n coefficients $\lambda_1, \lambda_2, \ldots\ldots \lambda_n$. These equations are of the form

$$A_1\lambda_1 + A_2\lambda_2 + \&c. = 0\,; \tag{5}$$

we must have, therefore,

$$\lambda_1 du_1 + \lambda_2 du_2 + \&c. = 0, \tag{6}$$

for all values of λ_1, λ_2, &c., which satisfy the $p-1$ equations (5). Hence, if we eliminate $p-1$ of these coefficients from (6) by means of the equations (5), we may equate separately to zero the coefficients of the rest. We shall then have $n-p+1$ equations of the form

$$L_1 du_1 + L_2 du_2 + \&c. = 0.$$

Some particular cases of the general proposition require to be noticed:

1. If there be but one equation of the form

$$L_1 du_1 + L_2 du_2 + \&c. = 0,$$

it is plain that this equation may always be satisfied by positive values of du_1, du_2, &c., unless the coefficients L_1, L_2, &c. have all the same sign.

2. The same is true even if there be more equations than one; provided that these equations are capable of being reduced to the form

$$\begin{aligned}
L_1 du_1 + L_2 du_2 + \&c. + L_i du_i &= 0, \\
L_{i+1} du_{i+1} + \&c. \qquad\qquad &= 0, \\
\&c. \qquad\qquad \&c.&
\end{aligned} \tag{7}$$

no two of these equations containing the same differential. In this case the coefficients of each of the equations must have the same sign.

3. If one of the variables, as x_1, enter into one only of the functions, as u_1, the given value a_1 must be an absolute maximum with regard to this variable.

For if we make

$$dx_2 = 0, \quad dx_3 = 0, \quad \&c. \; dx_m = 0,$$

we shall have

$$du_1 = \frac{du_1}{dx_1} dx_1, \; du_2 = 0, \; du_3 = 0, \; \&c. \; du_n = 0.$$

If now we take dx_1 of the same sign as $\frac{du_1}{dx_1}$, we shall have a positive value of du_1: unless, therefore,

$$\frac{du_1}{dx_1} = 0,$$

it will be possible to augment the value of u_1 without any augmentation of u_2, u_3, &c. In this case, then, we must have

$$\frac{du_1}{dx_1} = 0,$$

implying that the function u_1 is an absolute maximum with regard to x_1.

4. The foregoing Lemmas enable us at once to solve the question of Prop. I. For if we find that, for a given position of the system, the equations

$$u_1 = a_1, \; u_2 = a_2, \; \&c. \; u_k = a_k \tag{8}$$

can be satisfied by real values of R_1, R_2, &c.; and if, moreover, we have ascertained, by the method of Prop. IV., that the system (8) is an *extreme* system, then we infer that the given position of the material points may be an extreme position of equilibrium.

5. We shall proceed to consider a particular case which will elucidate the general principle.

Let the number of the geometrical forces R_1, R_2, &c., acting

at the points 1, 2, 3, ... k be the same as the number of the points. We shall have then

$$du_1 = \frac{du_1}{dR_1}\,dR_1 + \frac{du_1}{dR_2}\,dR_2 + \&c. + \frac{du_1}{dR_k}\,dR_k,$$

$$du_2 = \frac{du_2}{dR_1}\,dR_1 + \&c., \qquad (9)$$

$$\&c., \qquad \&c.,$$

$$du_k = \frac{du_k}{dR_1}\,dR_1 + \&c.$$

If all the geometrical forces act at every one of the points 1, 2, ... k; or, in other words, if the co-ordinates of all these points enter into each one of the equations of condition, the coefficients

$$\frac{du_1}{dR_1},\quad \frac{du_1}{dR_2}, \&c.\ \frac{du_2}{dR_1}, \&c.$$

will be all finite.

If any one of the forces, as R_1, only act at the points 1, 2, ... p, we shall have

$$\frac{du_{p+1}}{dR_1} = 0,\quad \frac{du_{p+2}}{dR_1} = 0, \ldots\ldots \frac{du_k}{dR_1} = 0.$$

We shall suppose as the most general case, that every geometrical force acts at every point. There is evidently no difficulty in introducing the limitations which result from the absence of any of these forces.

We have seen that if the points 1. 2, ... k be in an extreme position of equilibrium, the elimination of dR_1, dR_2, &c. among the equations (9), must give at least one equation of the form

$$L_1du_1 + L_2du_2 + \&c. + L_kdu_k = 0.$$

Hence, according to the theory of elimination among linear equations, we must have the determinant,

$$\begin{vmatrix} \frac{du_1}{dR_1}, & \frac{du_1}{dR_1}, & \cdots\cdots & \frac{du_k}{dR_k} \\ \frac{du_2}{dR_1}, & \cdots\cdots\cdots & & \\ \cdots\cdots & \cdots\cdots & & \\ \frac{du_k}{dR_1}, & \cdots\cdots\cdots & & \end{vmatrix} = 0. \qquad (10)$$

11. It may be well to investigate the geometrical meaning of any one of the differential coefficients,

$$\frac{du_1}{dR_1}, \quad \frac{du_1}{dR_2}, \text{ \&c. } \frac{du_2}{dR_1}, \text{ \&c.}$$

We have (p. 65)

$$u_1 = \frac{N_1 + R_2 \cos \overline{N_1R_1} + R_2 \cos \overline{N_1R_2} + \text{\&c.}}{V_1}$$

where

$$V_1^2 = F_1^2 + R_1^2 + R_2^2 + \text{\&c.} + \overline{2F_1R_1} \cos \overline{F_1R_1} + \text{\&c.}$$

$$+ 2F_1R_2 \cos \overline{F_1R_2} + \text{\&c.} + 2R_1R_2 \cos \overline{R_1R_2} + \text{\&c.}$$

Differentiating, we find

$$\frac{du_1}{dR_1} = \frac{\cos \overline{N_1R_1}}{V_1} - \frac{(N_1 + R_1 \cos \overline{N_1R_1} + \text{\&c.})(R_1 + F_1 \cos \overline{F_1R_1} + \text{\&c.})}{V_1^3}$$

But since V_1 is the resultant of all the forces acting at the point 1, it is plain that

$$N_1 + R_1 \cos \overline{N_1R_1} + \text{\&c.} = V_1 \cos \overline{V_1N_1}$$

$$R_1 + F_1 \cos \overline{F_1R_1} + \text{\&c.} = V_1 \cos \overline{V_1R_1}.$$

Substituting these values, we have

$$\frac{du_1}{dR_1} = \frac{1}{V_1} (\cos \overline{N_1R_1} - \cos \overline{V_1N_1} \cos \overline{V_1R_1})$$

Denoting by the symbol $\overline{N_1V_1R_1}$ the angle between the plane of N_1V_1, and the plane of V_1, R_1, we know that

$$\cos \overline{N_1 R_1} - \cos \overline{V_1 N_1} \cos \overline{V_1 R_1} = \sin \overline{V_1 N_1} \sin \overline{V_1 R_1} \cos \overline{N_1 V_1 R_1}.$$

Hence

$$\frac{du_1}{dR_1} = \frac{\sin \overline{V_1 N_1} \sin \overline{V_1 R_1} \cos \overline{N_1 V_1 R_1}}{V_1}. \qquad (11)$$

Similar expressions hold for the other differential coefficients.

If one of the geometrical forces R_1 act at one only of the points which are in an extreme position, it is plain from p. 73 that we must have

$$\frac{du_1}{dR_1} = 0.$$

Substituting the value given above, we must have one of the following conditions:—

$$\overline{V_1 N_1} = 0, \quad \overline{V_1 R_1} = 0, \quad \overline{N_1 V_1 R_1} = \frac{\pi}{2}.$$

The first of these equations would denote that the resultant of all the forces acting at the point 1 is directed along the normal. This is plainly impossible if the point be in an extreme position.

The second equation would denote that the force R_1 coincides in direction with the resultant of all the other forces. In this case it is plain that, so far as the points under consideration are concerned, the force R_1 is indeterminate. For the slipping of any point is determined by the direction, not the magnitude, of the force acting at that point. But for the point 1, since the direction of R_1 coincides with that of the resultant of all the other forces, it is plain that no change in the magnitude of R_1 will affect the direction of the resultant of all the forces acting at this point. And since R_1 does not act at any other of the points under consideration, none of the forces acting at these points will be affected in any way.

The third equation signifies that the plane passing through the complete resultant and the normal is at right angles to the plane passing through the same resultant and R_1. Hence, as the complete resultant is situated on the cone of resistance at the point 1, it is evident that the second of the above-mentioned planes is a tangent to this cone.

If only one point be in an extreme position, the value of u_1 must be an absolute maximum.

Example.

7. Three material points, A, B, C, rest, each on a rough surface, in given positions, and are acted on by given forces. The points A, C, are connected with B by rigid weightless rods, AB, CB. To determine whether any, and if so which, of the points are in extreme positions of equilibrium.

It is plain that, so far as the geometrical relations of the system are concerned, the equilibrium may be broken in any of the following ways:—

1. By the slipping of either or both of the extreme points A, C, the intermediate point B remaining unmoved.

2. By the simultaneous slipping of A and B, or C and B.

3. By the slipping of all three points A, B, C.

We shall proceed to consider the conditions to be fulfilled in each of these cases:—

(1.) If the point A alone be in an extreme position, it is plain that we must have

$$\frac{du_1}{dR_1} = 0.$$

It is plain from p. 76 that this is equivalent to supposing either that the acting force is directed along the line AB, and that this line is itself situated in the cone of resistance at the point A, or that the plane passing through AB and the direction of the acting force is a tangent to the cone of resistance at A. The line AB and the direction of the given acting force must, therefore, lie outside the cone of resistance.

Hence, if the acting force and the line AB be not coincident with the same side of the cone, the magnitude of the force R_1 may be determined as follows:—

Through AB draw a plane touching the cone of resistance; this plane must contain the direction of the acting force, and the line of contact will be the direction of the resultant V_1 of R_1 and F_1. Hence obviously

$$R_1 = F_1 \frac{\sin \overline{V_1 F_1}}{\sin \overline{V_1 R_1}}. \tag{12}$$

Let us next consider the conditions to be fulfilled at the

point B, which is supposed not to be in an extreme position of equilibrium.

This point is acted on by three forces—namely, 1, a given force, F_2; 2, a force $-R_1$ directed along the line AB; 3, a force $-R_2$ directed along the line CB. The second of these forces being determined by the equation (12), we may suppose B to be acted on by two forces—namely, 1, the resultant of F_2 and $-R_1$, which is a known force; 2, a force $-R_2$ situated in the line CB. Hence it is evident that the plane passing through CB and the resultant of F_2 and $-R_1$ must cut the cone of resistance at the point B, and that one at least of the lines of section must lie *within* the angle contained by $-R_2$ and this resultant.

Similarly the point C, which is acted on by the two forces F_3 and R_2, not being in an extreme position, it is evident that the plane containing the directions of these two forces must cut the cone of resistance at C, and that one at least of the sections must lie within the angle $\overline{F_3R_2}$.

Moreover, it must be possible to assign such a value to R_2 that the resultant of F_2, $-R_1$, and $-R_2$ may lie within the cone of resistance at B, and that the resultant of F_3 and R_2 may lie within the corresponding cone at C.

1. If the line of direction of the forces R_2, $-R_2$ lie within both the cones, this condition may always be fulfilled by taking R_2 sufficiently large.

2. Let the directions of the forces R_2, $-R_2$ be situated, the one without, and the other within the corresponding cone.

Draw a plane through $-R_2$ and the resultant of F_2 and $-R_1$. This plane will cut the cone of resistance at B in two lines, one, at least, of which will lie within the angle made by the directions of these forces. Let the angles which these directions make with this side of the cone be α, β, the latter being the angle corresponding to $-R_2$. Make a similar construction for the point C, and let the corresponding angles be α', β'. Suppose the angle β to be situated within the cone at B, and the angle β' to be situated without the cone at C; then, for the value $R_2 = \infty$, it is plain that the resultant of the forces at C would lie without the cone of resistance, and that the resultant at B would, for the same value, lie within the corresponding cone. If, now, con-

tinually diminishing values be assigned to R_2, the effect of this diminution will be to cause the resultant at B to move *outwards*, and the resultant at C *inwards*, with respect to the corresponding cones. Let P_2 be the resultant of F_2 and $-R_1$. Then, when R_2 attains the value

$$P_2 \frac{\sin a}{\sin (a + \beta)},$$

the resultant at B will in its outward motion have reached the surface of its cone. Again, when R_2 attains the value

$$F_3 \frac{\sin a'}{\sin (a' + \beta')},$$

the resultant at C in its inward motion will have reached the surface of its cone. It is then necessary and sufficient for the fulfilment of the condition which we are investigating, that the first of these values should be less than the second. Hence,

$$P_2 \frac{\sin a}{\sin (a + \beta)} < F_3 \frac{\sin a'}{\sin (a' + \beta')}.$$

3. If this line be without both these cones, and if the directions of the given forces lie within both cones,* the condition may always be satisfied by taking R_2 sufficiently small.

(2.) Next, let it be supposed that A and B are both in extreme positions, and that C is not in an extreme position. Then it is plain that, in accordance with the principle of p. 76, we must have

$$\frac{du_2}{dR_2} = 0. \qquad (13)$$

The conditions required are, therefore,

$$u_1 = a_1, \; u_2 = a_2, \; \frac{du_2}{dR_2} = 0.$$

The first two of these conditions are sufficient to determine R_1 and R_2. If then the points A, B, be in extreme positions, and if C be not in an extreme position, the substitution of the values

* If this be not true, the problem is solved as in (2).

so found must satisfy the remaining condition, and the value of R_2 must make $u_3 > a_3$.

It is easy to interpret these conditions geometrically. From the equation (13) we learn that, unless the line BC be situated on the surface of the cone of resistance at B, the plane passing through this line and the resultant of all the forces at B is a tangent to this cone, the line of contact being, of course, the direction of the resultant. This direction is, therefore, known. Hence, of the three forces, $F_2, -R_1, -R_2$, acting at the point B, we know the directions, the magnitude of one, and the direction of the resultant. These forces are, therefore, completely determined. If now we compound the forces R_1, F_1, acting at A, the resultant of these forces ought to be situated on the cone of resistance at A. Moreover, since

$$du_1 = \frac{du_1}{dR_1}\,dR_1 = \frac{1}{V_1}\sin\,\overline{V_1N_1}\,\sin\,\overline{V_1R_1}\,\cos\,\overline{N_1V_1R_1}\,dR_1$$

$$du_2 = \frac{du_2}{dR_1}\,dR_1 = \frac{1}{V_2}\sin\,\overline{V_2N_2}\,\sin\,\overline{V_2R_1}\,\cos\,\overline{N_2V_2R_1}\,dR_1 \qquad (14)$$

it is further necessary that, of the two dihedral angles $\overline{N_1V_1R_1}$, $\overline{N_2V_2R_1}$, one shall be acute and the other obtuse.*

(3.) Lastly, let it be supposed that all three points are in extreme positions; we must then have the following conditions—

$$u_1 = a_1,\ u_2 = a_2,\ u_3 = a_3. \qquad (15)$$

These equations must give real values for R_1 and R_2. Eliminating these two quantities from the three equations (15), we have an equation of condition between the given quantities of the question. Moreover, in the equations

$$du_1 = \frac{du_1}{dR_1}\,dR_1,\ du_2 = \frac{du_2}{dR_1}\,dR_1 + \frac{du_2}{dR_2}\,dR_2,\ du_3 = \frac{du_3}{dR_2}\,dR_2,$$

it must be impossible to assign any values to dR_1, dR_2, which

* It must be remembered that the directions of R_1 in these angles, respectively, are opposite. If we measure R_1 in the same direction, the condition requires that the affections of these angles shall be the same.

will make du_1, du_2, du_3 positive. Eliminating dR_1, dR_2 between these equations, we have

$$du_2 = \frac{\frac{du_2}{dR_1}}{\frac{du_1}{dR_1}} du_1 + \frac{\frac{du_2}{dR_2}}{\frac{du_3}{dR_2}} du_3.$$

Hence, it is plain that the coefficients of du_1, du_2, in this equation must both be negative. Interpreting this as before, we infer that the dihedral angles $\overline{N_1V_1R_1}$, $\overline{N_3V_3R_2}$ must have the same affections with the angles $\overline{N_2V_2R_1}$, $\overline{N_2V_2R_2}$ respectively, the directions of R_1, R_2, being in the last two angles the same as in the first two.

II.

Extreme positions of a solid body.

8. When a solid body, resting upon a number of others, is disturbed either by an external shock, or by the diminution of the coefficient of friction at one or more of the points of support, equilibrium will, in general, be broken simultaneously at all the points of support. This rupture of equilibrium may happen at each point in any one of three ways:—1. By slipping of one body upon the other. 2. By rolling without slipping—this will rarely happen unless both bodies be moveable. 3. By the separation of one body from the other at the point of contact.

In the problems with which we are concerned, it is in general necessary to the rupture of equilibrium that at some one or more of the points of contact motion of the first kind should take place. For, if motion consisting altogether of the second and third kinds were dynamically possible, it would have been equally possible *before* the diminution of the coefficient of friction; equilibrium could not, therefore, have existed. If, then, the body be capable of being set in motion by such a diminution, it must necessarily be about to *slip* at one or more of the points of contact. This we may express by saying that if the body itself be in an extreme position, some one or more of its points of contact must be in an extreme position, applying that expression to points which are characterized by a movement of

slipping, not to those which are characterized by a movement of rolling or of separation.

The criterion of points which are, in this sense, in extreme positions of equilibrium is to be investigated in a manner perfectly analogous to that which has been adopted for the case of a system of material particles. If the equations of equilibrium enable us to determine, at least in direction, the force of resistance at each point of contact, it is only necessary to inquire whether the obliquity of any of these directions to the corresponding normal be equal to, or less than the angle of friction. In the former case the point of contact is, and in the latter is not, in an extreme position of equilibrium.

If the equations of equilibrium be insufficient to determine these directions, the points of contact which are in extreme positions must be determined by conditions identical with those of p. 64, namely—

(1.) At every such point the angle of friction must be a *possible* value for the obliquity of the force of resistance.

(2.) It must be impossible to assign to the obliquity of the force of resistance at this point a value less than the angle of friction, without assigning to this obliquity at some other point an angle greater than the corresponding angle of friction.

The remainder of the investigation is also analogous to that of the former case. The cosine of each obliquity is given by the equations of equilibrium as a function of a number of variables, and it is necessary to consider whether it be possible to assign to these variables real values which will satisfy the equations

$$u_1 = a_1,\ u_2 = a_2,\ \&c.;$$

and also whether it be *im*possible to augment simultaneously the entire system of values, a_1, a_2, &c., without causing some of the variables to become imaginary.

Example.

9. A solid body, acted on by any given forces, rests upon three fixed rough surfaces. To determine the nature of the equilibrium, if the body be in an extreme position.

We shall suppose that the body actually rests upon the three surfaces, or, in other words, that the pressure at each point of contact is finite. We thus exclude from consideration all cases in which equilibrium is on the point of being broken by the *separation* of the heavy body from one or more of the supporting surfaces.

These cases being excluded, the rupture of equilibrium may take place in one of three ways—1. By slipping at one point of contact combined with rotation round the other two points. 2. By slipping at two points of contact combined with rotation round the third point. 3. By slipping at all three points. This is expressed in the phraseology of the present chapter by saying that one, two, or all of the points of contact may be in extreme positions.

With regard to the first of these, it is easy to show that, except in a particular case, the motion there supposed is geometrically impossible. Let A, B, C, be the points of contact, and let it be supposed that slipping takes place only at C. Then, since the motion is one of rotation round A, and also round B, it must be a motion of rotation round the axis AB. The motion at C must, therefore, be perpendicular to the plane ABC. But inasmuch as this motion is also a motion of slipping, its line of direction must be contained in the tangent plane at C. Hence it is evident that the motion here supposed is impossible, unless the normal to the supporting surface at C lie in the plane ABC.

It is easy to see, also, that the system admits in general of one definite motion of the second kind. For, if the motion be one of rotation round A, and of slipping at B and C, it must be possible to draw through A an axis of rotation AR, such that the planes ABR, ACR shall contain respectively the normals to the supporting surfaces at B and C. Conversely, if such a line *can* be drawn, the supposed motion is possible. But it is evident that the required line is obtained by drawing through A two planes passing through the normals at B and C respectively. The intersection of these planes is the required axis of rotation.

The third movement is evidently possible in an infinite variety of ways.

Proceeding now to consider the problem statically, we shall take the plane of the three points of contact as the plane of xy. The co-ordinates of the points of application of the three forces of resistance are then x_1y_1o, x_2y_2o, x_3y_3o. If now the components of these forces be $X_1Y_1Z_1$, $X_2Y_2Z_2$, $X_3Y_3Z_3$, and if we denote the components of the resultant of the acting forces by X, Y, Z, and the components of the resultant moment by L, M, N, we have the six equations of equilibrium,

$$\begin{gathered} X_1 + X_2 + X_3 = X, Y_1 + Y_2 + Y_3 = Y, \quad Z_1 + Z_2 + Z_3 = Z, \\ Z_1y_1 + Z_2y_2 + Z_3y_3 = L, \quad -Z_1x_1 - Z_2x_2 - Z_3x_3 = M, \qquad (16) \\ Y_1x_1 + Y_2x_2 + Y_3x_3 - X_1y_1 - X_2y_2 - X_3y_3 = N. \end{gathered}$$

Moreover, denoting by $\alpha_1\beta_1\gamma_1$, $\alpha_2\beta_2\gamma_2$, $\alpha_3\beta_3\gamma_3$, the direction cosines of the normals to the supporting surfaces at the points A, B, C, respectively; and by u_1, u_2, u_3, the cosines of the angles which these normals make respectively with the directions of the forces of resistance, we have

$$\begin{aligned} (\alpha_1X_1 + \beta_1Y_1 + \gamma_1Z_1)^2 &= u_1^2(X_1^2 + Y_1^2 + Z_1^2) \\ (\alpha_2X_2 + \beta_2Y_2 + \gamma_2Z_2)^2 &= u_2^2(X_2^2 + Y_2^2 + Z_2^2) \qquad (17) \\ (\alpha_3X_3 + \beta_3Y_3 + \gamma_3Z_3)^2 &= u_3^2(X_3^2 + Y_3^2 + Z_3^2). \end{aligned}$$

We have also

$$u_1 > \text{or} = \cos\epsilon_1, \quad u_2 > \text{or} = \cos\epsilon_2, \quad u_3 > \text{or} = \cos\epsilon_3,$$

ϵ_1, ϵ_2, ϵ_3 being the angles of friction at the three points of support. We now proceed to consider the several cases enumerated above.

(1.) Let it be supposed that only the point C is in an extreme position. Then we must have

$$u_1 > \cos\epsilon_1, \quad u_2 > \cos\epsilon_2, \quad u_3 = \cos\epsilon_3.$$

Moreover, since there is but one point in an extreme position, it is plain that, in accordance with the principles stated in p. 76, this value of u_3 must be an absolute maximum. Now, it appears, from the third, fourth, and fifth of the equations (16), that Z_1, Z_2, Z_3 are determinate; and from the remaining three equations (16) that of the remaining six components X_1, X_2, X_3,

Y_1, Y_2, Y_3, three remain indeterminate. Hence, we have u_3 given by the third equation (17) as a function of the two independent variables X_3, Y_3. We must have, therefore,

$$\frac{du_3}{dX_3} = 0, \quad \frac{du_3}{dY_3} = 0.$$

Hence, from the third equation (17),

$$\begin{aligned}(a_3X_3 + \beta_3Y_3 + \gamma_3Z_3)\, a_3 &= u_3^2X_3 \\ (a_3X_3 + \beta_3Y_3 + \gamma_3Z_3)\, \beta_3 &= u_3^2\, Y_3.\end{aligned} \tag{18}$$

But we have seen (p. 83) that in this case the normal at C must lie in the plane ABC. We must have, therefore, $\gamma_3 = 0$. Hence, squaring and adding the equations (18), we obtain, without difficulty, from equation 3 (17),

$$X_3^2 + Y_3^2 + Z_3^2 = u_3^2\,(X_3^2 + Y_3^2),$$

an impossible equation, inasmuch as $u_3 < 1$. It is therefore impossible that one only of the points of contact should be in an extreme position.

(2.) Let it be supposed that B and C are both in extreme positions. We have then $u_2 = \cos \varepsilon_2$, $u_3 = \cos \varepsilon_3$. Differentiating now the second and third of equations (17), recollecting that Z_2 and Z_3 are constant, and putting

$$\begin{aligned}R_2^2 &= X_2^2 + Y_2^2 + Z_2^2, \quad R_2u_2 = a^2X_2 + \beta_2Y_2 + \gamma_2Z_2 \\ R_3^2 &= X_3^2 + Y_3^2 + Z_3^2, \quad R_3u_3 = a_3X_3 + \beta_3Y_3 + \gamma_3Z_3,\end{aligned}$$

we have

$$\begin{aligned}R_2^2du_2 &= R_2\,(a_2dX_2 + \beta_2dY_2) - u_2(X_2dX_2 + Y_2dY_2) \\ R_3^2du_3 &= R_3\,(a_3dX_3 + \beta_3dY_3) - u_3\,(X_3dX_3 + Y_3dY_3).\end{aligned} \tag{19}$$

Now, if B, C, be in extreme positions, it must be impossible to assign to dX_2, dY_2, dX_3, dY_3 such values as will render du_2, du_3 both positive. Hence, in accordance with the principle established in p. 70, there must be, independently of these differentials, an equation of the form

$$L_2du_2 + L_3du_3 = 0,$$

in which L_2, L_3 have the same sign. Eliminating X_1, Y_1 from

the first, second, and sixth of equations (16), and differentiating the resulting equation, we have

$$(y_2 - y_1)\,dX_2 + (y_3 - y_1)\,dX_3 - (x_2 - x_1)\,dY_2 - (x_3 - x_1)\,dY_3 = 0.$$

If, then, we add to this the first equation (19) multiplied by L_2, and the second multiplied by L_3, we may equate separately to zero the coefficients of dX_2, dX_3, dY_2, dY_3. We have then

$$\left.\begin{array}{l} L_2(\alpha_2R_2 - u_2X_2) + y_2 - y_1 = 0,\ L_2(\beta_2R_2 - u_2Y_2) - (x_2 - x_1) = 0 \\ L_3(\alpha_3R_3 - u_3X_3) + y_3 - y_1 = 0,\ L_3(\beta_3R_3 - u_3Y_3) - (x_3 - x_1) = 0. \end{array}\right\} \quad (20)$$

Eliminating L_2, L_3 from these equations, we find

$$\left.\begin{array}{l} R_2\{\alpha_2(x_2 - x_1) + \beta_2(y_2 - y_1)\} = u_2\{(x_2 - x_1)X_2 + (y_2 - y_1)Y_2\} \\ R_3\{\alpha_3(x_3 - x_1) + \beta_3(y_3 - y_1)\} = u_3\{(x_3 - x_1)X_3 + (y_3 - y_1)Y_3\} \end{array}\right\} \quad (21)$$

We have thus altogether ten equations—namely, the six equations (16); the second and third of equations (17), and the two equations (21), to determine the nine unknown quantities, $X_1\,Y_1\,Z_1$, $X_2\,Y_2\,Z_2$, $X_3\,Y_3\,Z_3$. These quantities being eliminated, there remains a single equation of condition between the given quantities of the problem which must be satisfied if the points B, C are in extreme positions, the point A not being in an extreme position. It is further necessary that the values of L_2, L_3, obtained from equations (20), should have the same sign.

To interpret the equations (21), let the symbols $\overline{N_2B}$, $\overline{R_2B}$ denote the angles which the line AB makes respectively with the normal and the force of resistance at B. Then

$$\alpha_2(x_2 - x_1) + \beta_2(y_2 - y_1) = AB\cos\overline{N_2B}$$

$$(x_2 - x_1)X_2 + (y_2 - y_1)Y_2 = R_2 \times AB\cos\overline{R_2B}.$$

Substituting these values in the first equation (21), we find

$$\cos\overline{N_2B} = \cos\overline{N_2R_2}\cos\overline{R_2B}.$$

Hence it appears that the planes of the angles $\overline{N_2R_2}$, $\overline{R_2B}$ are at right angles to each other. Since, then, the direction of R_2 lies upon the cone of resistance of which N_2 is the axis, it is

evident that this direction is found by drawing through AB a tangent plane to the cone of resistance at B, the line of contact being the direction required. A similar interpretation holds for the second equation (21). If, therefore, either of the lines AB, AC lie within the cone of resistance at B or C, respectively, this point cannot be in an extreme position.

With regard to the condition which requires that L_2, L_3 should have the same sign, it is evident that the origin may be so assumed that $y_2 - y_1$ and $y_3 - y_1$ shall both be positive. Hence, from the first and third of equations (20), it appears that the signs of L_2 and L_3 depend upon the signs of the quantities

$$a_2 R_2 - u_2 X_2, \quad a_3 R_3 - u_3 X_3;$$

or, using the foregoing notation to express the angles, upon the signs of the quantities

$$\cos \overline{N_2 X} - \cos \overline{R_2 N_2} \cos \overline{R_2 X}, \qquad \cos \overline{N_3 X} - \cos \overline{R_3 N_3} \cos \overline{R_3 X}.$$

To interpret this, let us conceive a sphere to be described round A, and the several lines and planes to be projected on it by parallel lines and planes through A. This projection is represented in the accompanying figure (Fig. 18), in which B, C

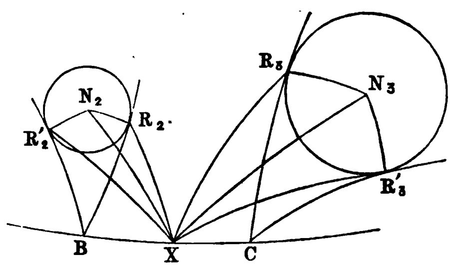

Fig. 18.

are the projections of the lines AB, AC; N_2, N_3 the projections of the normals, and the small circles the projections of the cones of resistance. Then, if BR_2, BR'_2, CR_3, CR'_3 be tangents to the small circles, it is evident, from the above, that the projection of the force of resistance R_2 must be either R_2 or R'_2, and the projection of R_3 either R_3 or R'_3.

Let X be the projection of the axis of x. Then

$$\cos \overline{N_2X} - \cos \overline{R_2N_2} \cos \overline{R_2X} = \sin R_2N_2 \sin R_2X \cos N_2R_2X,$$
$$\text{or} = \sin R'_2N_2 \sin R'_2X \cos N_2R'_2X.$$

Hence it is evident that the sign of L_2 depends upon the affection of the angle N_2R_2X or $N_2R'_2X$. But it is easily seen that of these two angles one is always obtuse, and the other acute; the same being, of course, true of the angles N_3R_3X, $N_3R'_3X$. In order, therefore, that L_2, L_3 may have the same sign, the forces of resistance must either correspond to the points R_2, R_3, or R'_2, R'_3.

(3.) Let it be supposed that the three points A, B, C are all in extreme positions. We have then $u_1 = \cos \varepsilon_1$, $u_2 = \cos \varepsilon_2$, $u_3 = \cos \varepsilon_3$. Differentiating the three equations (17), recollecting that Z_1, Z_2, Z_3 are constant, we have

$$\left.\begin{aligned} R_1^2 du_1 &= R_1(\alpha_1 dX_1 + \beta_1 dY_1) - u_1(X_1dX_1 + Y_1dY_1) \\ R_2^2 du_2 &= R_2(\alpha_2 dX_2 + \beta_2 dY_2) - u_2(X_2dX_2 + Y_2dY_2) \\ R_3^2 du_3 &= R_3(\alpha_3 dX_3 + \beta_3 dY_3) - u_3(X_3dX_3 + Y_3dY_3). \end{aligned}\right\} \quad (22)$$

Again, differentiating the first, second, and sixth of equations (16), we have

$$\left.\begin{aligned} dX_1 + dX_2 + dX_3 = 0,\ dY_1 + dY_2 + dY_3 = 0, \\ y_1dX_1 + y_2dX_2 + y_3dX_3 = X_1dY_1 + x_2dY_2 + x_3dY_3. \end{aligned}\right\} \quad (23)$$

Then, if all the points of contact be in extreme positions, it must be impossible to assign to the increments dX_1, dX_2, &c., such values that du_1, du_2, du_3 may be all positive. Hence, in accordance with the principle stated in p. 70, there must exist, for all values of these increments consistent with the equations (22), an equation of the form

$$L_1du_1 + L_2du_2 + L_3du_3 = 0,$$

in which L_1, L_2, L_3 have all the same sign. Hence, if we multiply the equations (22) and (23), respectively, by L_1 L_2, L_3, P, Q, and 1, and add them, we may equate separately to zero the coefficients of the differentials dX_1, &c. We have then

$$L_1 (R_1\alpha_1 - u_1X_1) + P + y_1 = 0, \quad L_2 (R_2\alpha_2 - u_2X_2) + P + y_2 = 0,$$

$$L_3 (R_3\alpha_3 - u_3X_3) + P + y_3 = 0,$$

$$L_1 (R_1\beta_1 - u_1 Y_2) + Q - x_1 = 0, \quad L_2 (R_2\beta_2 - u_2 Y_2) + Q - x_2 = 0,$$

$$L_3 (R_3\beta_3 - u_3 Y_3) + Q - x_3 = 0.$$

Eliminating $L_1\ L_2\ L_3\ P\ Q$ from these equations, we have a single equation of condition which must be satisfied by the values of $X_1\ X_2\ X_3$, $Y_1\ Y_2\ Y_3$, $Z_1\ Z_2\ Z_3$ which are obtained from the nine equations (16), and (17). Moreover, the values of $L_1\ L_2\ L_3$ obtained from the foregoing equations must have the same sign. These are the conditions which must necessarily be fulfilled, if the three points of contact be in extreme positions.

10. We have thus obtained for a system of particles and for a solid body the conditions which must be fulfilled, if any given point, or system of points, be in an extreme position. But we cannot infer conversely that, if these conditions *be* fulfilled, the given system of points, or body, is necessarily in an extreme position. It is true that, *if* the forces which are not determined by the equations of equilibrium have the values assigned by these conditions, a diminution in the coefficient of friction at any of the points under consideration would cause a rupture of equilibrium. But we cannot say that the forces which are undetermined by the equations of equilibrium must necessarily have the values corresponding to an extreme position of equilibrium. These forces may have any system of values consistent with equilibrium, the particular system which obtains in any particular case depending not only upon the position of the particles or body and the forces which act upon them, but also upon other conditions, which may be termed "initial;" namely, the forces which have been used in placing the particles or body in the given position. We shall consider this point more fully afterwards; for the present it is sufficient to remark that the importance of these conditions will at once appear, if we state the question of equilibrium as it really exists in nature, rejecting the abstractions of Rational Mechanics, and replacing the forces which we have called "geometrical" by the elasticity of the

rods, strings, &c., by which geometrical conditions are mechanically realized. In this way of considering the question, it becomes evident that two systems, identical in position and in the external forces which act upon them, may, nevertheless, differ widely in the values of the forces of elasticity exerted by these connecting rods, strings, &c. These forces will, within certain limits, depend upon the states of compression or extension in which these rods and strings have been originally placed, or, in other words, upon the conditions which we have called "initial."

CHAPTER IV.

MOVEMENT OF A PARTICLE, OR SYSTEM OF PARTICLES.

I.—*Movement of a Single Particle upon a Fixed Rough Surface.*

1. We have seen (p. 7) that the force of friction, when acting on a particle in motion, has necessarily its greatest value, and is always directly opposed to the movement of the particle. Hence, if x, y, z be the co-ordinates of the particle, ds the element of the trajectory, α, β, γ the direction cosines of the normal to the supporting surface, and X, Y, Z the components of the acting force, the equations of motion are

$$\frac{d^2x}{dt^2} = X + N\left(\alpha - \mu\frac{dx}{ds}\right), \quad \frac{d^2y}{dt^2} = Y + N\left(\beta - \mu\frac{dy}{ds}\right),$$
$$\frac{d^2z}{dt^2} = Z + N\left(\gamma - \mu\frac{dz}{ds}\right); \qquad (1)$$

N being the normal reaction of the surface. The unknown quantities are evidently the same as in the problem of the motion of a particle upon a smooth surface—namely, x, y, z, N. The equations by which these unknown quantities are determined are also similar to the equations of the other problem—namely, the three equations of motion, and the equation of the supporting surface. These equations are rarely integrable, even in the simplest cases, but some general properties of the movement may be deduced from them.

2. If the independent variable be changed from t to s, the foregoing equations will become

$$v^2\frac{d^2x}{ds^2} + v\frac{dv}{ds}\frac{dx}{ds} = X + N\left(\alpha - \mu\frac{dx}{ds}\right), \quad v^2\frac{d^2y}{ds^2} + \&c. = Y + \&c.,$$
$$v^2\frac{d^2z}{ds^2} + \&c. = Z + \&c. \qquad (2)$$

Let a, b, c be the direction cosines of a line, L, in space, and

let these equations be multiplied respectively by a, b, c, and added. We have then,

$$v^2\left(a\frac{d^2x}{ds^2}+b\frac{d^2y}{ds^2}+c\frac{d^2z}{ds^2}\right)+\left(v\frac{dv}{ds}+\mu N\right)\left(a\frac{dx}{ds}+b\frac{dy}{ds}+c\frac{dz}{ds}\right)$$
$$= aX + bY + cZ + N(a\alpha + b\beta + c\gamma). \tag{3}$$

(1.) Let it be supposed that L is perpendicular to the plane of the normal section passing through the tangent to the trajectory. Then

$$a\frac{dx}{ds}+b\frac{dy}{ds}+c\frac{dz}{ds}=0, \quad a\alpha + b\beta + c\gamma + 0.$$

Also, if ϕ be the angle which the osculating plane to the trajectory makes with the plane of the normal section, and θ the angle which the acting force, R, makes with the latter plane, we have, ρ being the radius of curvature of the trajectory,

$$-\frac{\sin\phi}{\rho}=a\frac{d^2x}{ds^2}+b\frac{d^2y}{ds^2}+c\frac{d^2z}{ds^2}, \quad R\sin\theta = aX + bY + cZ.$$

If these values be substituted in equation (3), it becomes

$$\frac{v^2}{\rho}\sin\phi + R\sin\theta = 0. \tag{4}$$

Hence, if $\theta = 0$, or, in other words, if the acting force lie in the plane of the normal section, we have $\phi = 0$, denoting that the osculating plane is normal to the surface, which is the well-known property of a geodetic line. This includes the case of motion in consequence of a primitive impulse only, without any acting force, as also the case in which the acting force is tangential; as, for example, when the particle moves in consequence of a primitive impulse in a resisting medium. In both these cases the trajectory will be a geodetic line.

(2.) Let L be perpendicular to the osculating plane to the trajectory. Then it is easy to see that the whole of the left-hand side of the equation (3) disappears, and that we have simply

$$aX + bY + cZ + N(a\alpha + b\beta + c\gamma) = 0, \tag{5}$$

denoting that the resolved part of the normal reaction perpendicular to the osculating plane is equal and opposite to the resolved part of the acting force along the same line; or, in other words, that the total force lies in the osculating plane. This is known to be a general property of curvilinear motion.

(3.) Let L be the normal to the surface; we have then

$$a\frac{dx}{ds} + b\frac{dy}{ds} + c\frac{dz}{ds} = 0, \quad a\frac{d^2x}{ds^2} + b\frac{d^2y}{ds^2} + c\frac{d^2z}{ds^2} = -\frac{\cos\phi}{\rho} = -\frac{1}{\rho^1},$$

where ρ^1 is the radius of curvature of the normal section. Also, $a\alpha + b\beta + c\gamma = 1$. Hence equation (3) becomes

$$\frac{v^2}{\rho^1} + R\cos\omega + N = 0 \qquad (6)$$

(where ω denotes the *obliquity*, or inclination to the normal, of the acting force)—showing that the *dynamical* pressure on the surface is equal to the square of the velocity divided by the radius of curvature of the normal section which passes through the tangent to the trajectory.

The equations (4), (5), (6), not containing the coefficient of friction explicitly, are consequently the same as they would be for a smooth surface. It is evident, then, that these three equations are not independent of each other. This might have been anticipated from the method in which these equations are obtained. For the three positions of the line L, by whose direction cosines the equations (2) are multiplied in order to obtain the equations (4), (5), (6), lie in the same plane—namely, the normal plane to the trajectory. And although the equations

$$aA + bB + cC = 0; \; a'A + b'B + c'C = 0; \; a''A + b''B + c''C = 0,$$

are in general equivalent to

$$A = 0, \quad B = 0, \quad C = 0;$$

it is known that this is not the case, if the three lines whose direction cosines are proportional respectively to a, b, c, a', &c., lie in the same plane.

(4.) Let L be the tangent to the trajectory. Then

$$a\frac{d^2x}{ds^2}+b\frac{d^2y}{ds^2}+c\frac{d^2z}{ds^2}=0,\quad a\frac{dx}{ds}+b\frac{dy}{ds}+c\frac{dz}{ds}=1,$$

$$a\alpha+b\beta+c\gamma=0.$$

Equation (3) becomes, therefore,

$$v\frac{dv}{ds}+\mu N=X\frac{dx}{ds}+Y\frac{dy}{ds}+Z\frac{dz}{ds}=\frac{dV}{ds},\qquad(7)$$

(if the acting force satisfy the condition

$$Xdx+Ydy+Zdz=dV).$$

We have then, by integration,

$$v^2=2V-2\int\mu Nds+\text{const}=2V-2\int Fds+\text{const.},\qquad(8)$$

where F is the force of friction. This is in accordance with the general principle according to which the *vis viva* is equal to double the work done by the acting forces. For it is evident that $-\int Fds$ is the work done by the force of friction.

3. As an example of the movement of a particle on a rough surface, let the supporting surface be a plane, and let us suppose gravity to be the only acting force. Let the supporting plane be inclined to the horizon at an angle α. Taking the axis of x parallel to the horizontal trace of the plane, and the axis of y in this plane perpendicular to the axis of x, we have for the normal and tangential forces, respectively, to the trajectory in the plane,

$$N=g\sin\alpha\frac{dx}{ds},\quad T=g\sin\alpha\frac{dy}{ds}-\mu g\cos\alpha.$$

Hence, by the ordinary formulæ,

$$\frac{v^2}{\rho}=g\sin\alpha\frac{dx}{ds},\quad v\frac{dv}{ds}=g\sin\alpha\frac{dy}{ds}-\mu g\cos\alpha.\qquad(9)$$

Integrating the second of these equations, we have

$$v^2=v'^2+2g(y\sin\alpha-\mu s\cos\alpha)=v'^2+2gy\sin\alpha\left(1-\frac{\mu s}{y}\cot\alpha\right),\qquad(10)$$

where v' is the velocity of projection. Now, if the angle of in-

clination be less than the angle of friction, we have $\mu \cot a > 1$, and, therefore, since $s > y$,

$$1 - \frac{\mu s}{y} \cot a = - A^2,$$

where A is finite. Hence the velocity is constantly diminishing, and the particle will come to rest in a finite space.

To find the equation of the trajectory, let

$$dy = pdx, \quad d^2y = qdx^2, \quad d^3y = rdx^3.$$

We have then

$$\frac{1}{\rho} = \frac{q}{(1 + p^2)^{\frac{3}{2}}}$$

Substituting for ρ and v^2 in the first equation (9), we have

$$v'^2 + 2g\,(y \sin a - \mu s \cos a) = g \sin a \left(\frac{1 + p^2}{q}\right). \qquad (11)$$

Differentiating and reducing, we find

$$2\mu q^2 \cot a = r\sqrt{1 + p^2}.$$

Now, since evidently

$$r = q\,\frac{dq}{dp},$$

we have, putting, for the sake of brevity, $n = 2\mu \cot a$,

$$nqdp = \sqrt{1 + p^2}\, dq.$$

Hence, by integration,

$$q = c\,(p + \sqrt{1 + p^2})^n, \qquad (12)$$

c being an arbitrary constant. Assume

$$\theta = \sqrt{1 + p^2} + p.$$

We have then

$$p = \frac{1}{2}\left(\theta - \frac{1}{\theta}\right), \quad q = \frac{dp}{dx} = \frac{1}{2}\left(1 + \frac{1}{\theta^2}\right)\frac{d\theta}{dx}.$$

Substituting this value in (12), we have

$$\frac{1}{2}\left(1 + \frac{1}{\theta^2}\right) d\theta = c\theta^n dx.$$

Integrating,

$$2cx + c' = \frac{\theta^{1-n}}{1-n} - \frac{\theta^{-(1+n)}}{1+n}. \qquad (13)$$

We have also

$$dy = pdx = \frac{1}{4c}\left(\theta^2 - \frac{1}{\theta^2}\right)\theta^{-(n+1)}\, d\theta.$$

Integrating,

$$4cy + c'' = \frac{\theta^{2-n}}{2-n} + \frac{\theta^{-(2+n)}}{2+n}. \qquad (14)$$

The trajectory is represented by the system of equations (13) and (14), between which θ is to be eliminated.

If $\tan \alpha > \mu$, whence $n < 2$, it is plain from equation (14) that when $y = \infty$, $\theta = \infty$, whence $p = \infty$, or the trajectory is ultimately perpendicular to the horizontal trace of the plane, supposed unlimited.

If $n > 1 < 2$, we have, for $p = \infty$, $2cx + c' = 0$; hence the ultimate value of x is finite, or the trajectory has an asymptot perpendicular to the horizontal trace of the plane. If $n < 1$, the curve has no asymptot.

The three arbitrary constants c, c', c'' are readily determined from the initial position, direction, and velocity of the projection. Thus, for example, if the particle be projected horizontally with a velocity v', and if we take the origin at the point of projection, we find from (11), (13), and (14),

$$c = \frac{g \sin \alpha}{v'^2}, \quad c' = \frac{2n}{1 - n^2}, \quad c'' = \frac{4}{4 - n^2}.$$

Integrating the equation,

$$ds = \sqrt{1 + p^2}dx = \frac{1}{4c}\left(\theta + \frac{1}{\theta}\right)^2 \theta^{-(n+1)}\, d\theta,$$

and determining the arbitrary constant from the conditions $s = 0$, $\theta = 1$, we find

$$4cs = \frac{\theta^{2-n}}{2-n} - \frac{\theta^{-(2+n)}}{2+n} - \frac{2\theta^{-n}}{n} - \frac{2n}{4-n^2} + \frac{2}{n} \qquad (15)$$

Substituting the values of y and s in equation (10), we find without difficulty,

$$v = \frac{1}{2} v' \left(\theta + \frac{1}{\theta} \right) \theta^{-\mu \cot a}. \qquad (14)$$

If $\mu \cot a > 1$, we have already seen that the final velocity $= 0$. If $\mu \cot a < 1$, it is plain from the foregoing equation that the final velocity is infinite. If $\mu \cot a = 1$, the same equation shows that the final value of v is half the velocity of projection.

To determine whether, in general, the velocity increase or diminish after projection, we recur to the second equation (9), from which we have

$$2vdv = g \sin a\,(2dy - nds).$$

But from equations (12) and (13), we have

$$4c\,(2dy - nds) = g \sin a \theta^{-(n+1)} \{ (2-n)\theta^2 - (2+n)\theta^{-2} - 2n \} \,d\theta.$$

The limit of the values of θ which make this positive is found by equating it to zero. This will give

$$\theta^2 = \frac{2+n}{2-n}, \quad \text{whence } p = \frac{n}{\sqrt{4-n^2}}.$$

If ϕ be the angle which the tangent to the trajectory makes with the horizontal trace of the plane, this equation gives $\sin \phi = \mu \cot a$. Hence we infer that the velocity will diminish until $\sin \phi = \mu \cot a$. If, therefore, the particle be projected *downwards* at an angle greater than this, it will continually increase. If it be projected *upwards*, or downwards at an angle less than this, it will at first diminish, and after attaining a minimum, continually increase. It is evident from the value of p, that this can happen only when $n < 2$. This agrees with what we have seen before (p. 95).

II.—*Movement of a System of Particles under the influence of Friction.*

Prop. I.

4. A system of material particles, each of which rests upon a rough surface, is connected by certain geometrical relations, and acted on by given forces. To determine the equations of motion.

Let $x_1 y_1 z_1$, $x_2 y_2 z_2$, $x_n y_n z_n$ be the co-ordinates of the particles, and $\alpha_1 \beta_1 \gamma_1$, $\alpha_2 \beta_2 \gamma_2$, . . . the direction cosines of the normals to the several supporting surfaces. Let, also, R_1, R_2, &c., be the normal reactions of these surfaces, and $X_1 Y_1 Z_1$, $X_2 Y_2 Z_2$, ... the components of the acting forces. Let

$$L = 0, \quad M = 0, \text{ \&c.},$$

be the equations expressing the geometrical relations which subsist between the particles of the system. Then, if we denote by $X'_1 Y'_1 Z'_1$, $X'_2 Y'_2 Z'_2$, the components of the several forces of friction, the equations of motion will be

$$
\left.
\begin{aligned}
m_1 \frac{d^2x_1}{dt^2} &= X_1 + X'_1 + \alpha_1 R_1 + \lambda_1 \frac{dL_1}{dx_1} + \lambda_2 \frac{dL_2}{dx_1} + \text{\&c.} \\
m_1 \frac{d^2y_1}{dt^2} &= Y_1 + Y'_1 + \beta_1 R_1 + \lambda_1 \frac{dL_1}{dy_1} + \lambda_2 \frac{dL_2}{dy_1} + \text{\&c.} \\
m_1 \frac{d^2z_1}{dt^2} &= Z_1 + Z'_1 + \gamma_1 R_1 + \lambda_1 \frac{dL_1}{dz_1} + \lambda_2 \frac{dL_2}{dz_1} + \text{\&c.} \\
m_2 \frac{d^2x_2}{dt^2} &= X_2 + X'_2 + \text{\&c., \&c.}
\end{aligned}
\right\} \quad (15)
$$

In addition to these equations, which are $3n$ in number, we have the n equations of the supporting surfaces, and the geometrical equations $L = 0$, $M = 0$, &c. The number of these equations is the same as that of the co-ordinates $x_1 y_1 z_1$, $x_2 y_2 z_2$, ... $x_n y_n z_n$; the normal reactions R_1, R_2, ... R_n, and the coefficients $\lambda_1 \lambda_2$, ... There remain, however, still to be determined the $3n$ components of the forces of friction, $X'_1 Y'_1 Z'_1$, ... $X'_n Y'_n Z'_n$. We require, therefore, $3n$ more equations

for the complete solution of the problem. These equations will differ according as each particle is in motion or at rest.

(1.) Let us suppose that the particle $x_1 y_1 z_1$ is in motion. Then we know that the force of friction acting upon this particle has its maximum value, and that it is directed in the tangent to the path of the particle. Let μ_1 be the coefficient of friction, and ds_1 the element of the path. Then we have

$$X'_1 = -\mu_1 R_1 \frac{dx_1}{ds_1},\ Y'_1 = -\mu_1 R_1 \frac{dy_1}{ds_1},\ Z'_1 = -\mu_1 R_1 \frac{dz_1}{ds_1}, \qquad (16)$$

and three similar equations for every particle which is in motion. Thus, if every particle were in motion, we should have the $3n$ additional equations which are necessary for the solution of the problem.

(2.) Let the particle $x_1 y_1 z_1$ be at rest. Then the first three equations become

$$\begin{aligned} &X_1 + X'_1 + \alpha_1 R_1 + \lambda_1 \frac{dL_1}{dx_1} + \lambda_2 \frac{dL_2}{dx_1} + \&c. = 0. \\ &Y_1 + Y'_1 + \beta_1 R_1 + \&c. = 0 \qquad (17) \\ &Z_1 + Z'_1 + \gamma_1 R_1 + \&c. = 0. \end{aligned}$$

We have besides, since the force of friction always acts in the tangent plane to the supporting surface,

$$\alpha_1 X'_1 + \beta_1 Y'_1 + \gamma_1 Z'_1 = 0. \qquad (18)$$

Now, since $x_1 y_1 z_1$ retain their original values, they are no longer unknown quantities. Hence, *if the equations of motion of the other particles of the system be sufficient to determine the coefficients* λ_1, λ_2, &c., the four equations just obtained will determine the four unknown quantities X'_1, Y'_1, Z'_1, R_1, and the solution will be complete. It remains to consider when the foregoing condition is fulfilled.

Let p of the particles be at rest, and let it be supposed that one of the geometrical equations, as $L_1 = 0$, only contains the co-ordinates of quiescent particles, *or that it is possible to form such an equation by eliminating the co-ordinates of the moving*

particles among the equations of condition. Then it is evident that the $4p$ equations (which are, in fact, equations of equilibrium) corresponding to these p particles will contain $4p + 1$ unknown quantities—namely, the $4p$ quantities,

$$X'_1\ Y'_1\ Z'_1\ R_1\ .\ .\ .\ .\ .\ .\ X'_p\ Y'_p\ Z'_p\ R_p,$$

and the coefficient λ_1, none of which will enter into any other equation. One of these quantities, therefore, must remain indeterminate. The mathematical reason of this is, as has been stated before, that the equation $L_1 = 0$, involving only quantities which are, in the present case, known quantities, gives us no additional information; while, at the same time, it necessarily introduces the new unknown quantity λ_1.

The same conclusion will, of course, hold, even if the co-ordinates of the moving particles enter into all the geometrical equations; provided that it be possible to form, by eliminating these co-ordinates, an equation or equations between the co-ordinates of the quiescent particles. For in these cases the number of the geometrical equations, considered as the means of determining unknown quantities, is reduced by the number of the equations which can be so formed, while the number of the unknown coefficients which are introduced by these geometrical equations remains the same as before.

5. We now proceed to show that the incompleteness of the solution, when it exists, only applies to the quiescent particles, and that for the particles in motion there are sufficient equations to determine their positions at any time, and also the forces, geometrical and frictional, which act upon them.

Suppose that k of the geometrical equations only involve the co-ordinates of quiescent particles. Then, as we have seen, these k equations, involving only known quantities, give us no assistance in determining the unknown quantities, while, at the same time, they introduce the k new unknown quantities, $\lambda_1, \lambda_2, \ldots \lambda_k$, which must, therefore, remain indeterminate. But it must be observed that no one of these coefficients enters into any of the equations which concern the moving particles. Every coefficient which appears in these equations has been introduced

by an equation involving the co-ordinates of one or more *moving* particles, which, being necessarily an equation between unknown quantities, does assist in the determination of these quantities.

6. The mathematical result of the whole discussion may be thus stated.

Every geometrical equation involving *only* the co-ordinates of quiescent particles augments the number of the unknown quantities by one, while it leaves the equations for the determination of these quantities precisely as it was before.

Every geometrical equation involving the co-ordinates of moving particles augments the number of the unknown quantities by one, but augments also by one the equations by which these quantities are determined, thus leaving the number of the unknown quantities unchanged. We have then, in the case of the moving particles, a sufficient number of equations for the solution of the problem—namely, 1. The dynamical equations, which are the same in number as the co-ordinates; 2. The equations of the supporting surfaces, which are the same in number with the normal reactions; 3. The equations (16) which are the same in number with the components of the forces of friction; 4. The geometrical equations, which are the same in number with the coefficients λ.

7. The physical result of the investigation is as follows:—

When a system of material particles, each of which rests upon a rough surface, is subject to the action of external forces, it will, in general, be found that of these particles some will be in a state of motion, and others in a state of rest.

Everything connected with the moving particles—namely, their positions, their velocities, and the forces, geometrical and frictional, which act upon them, is fully determined by means of the dynamical and geometrical equations.

The geometrical and frictional forces which act upon the quiescent particles, will also be determinate, unless it be possible to form by elimination one or more equations between the co-ordinates of the quiescent particles *only*. If this *be* possible, the geometrical force replacing every such equation will be indeterminate in intensity.

8. It remains now to consider whether we have any means of determining at any moment which of the particles are in motion. Let it be supposed that the initial velocities of the several particles are given, *and that no one of these velocities is absolutely zero*. Then the equations of the problem are, for every particle, those which correspond to a state of motion; and therefore, as we have seen, sufficient for the determination of all the unknown quantities. Suppose that these equations have been integrated, so as to determine the several co-ordinates as functions of the time. We have then, by differentiation, the velocities of the several particles at any given instant. And so long as these velocities continue finite, the solution obtained continues to be applicable.

Suppose now that the velocity of the point $x_1 y_1 z_1$ becomes at any instant zero. This particle being for the moment at rest, we must consider, in the first place, whether or not it will continue at rest. In order that this may be the case, the equations (17) (p. 99) must be satisfied.

Now, unless there be (independently of the equation of the supporting surface) a geometrical equation involving only the co-ordinates $x_1 y_1 z_1$, it is plain, from what has been said, that the coefficients λ_1, λ_2, &c., are all determined, as functions of the time, by the other equations of the system. Then the four equations (17) and (18) (p. 99) are sufficient to determine the four quantities $X'_1\ Y'_1\ Z'_1$, R_1. If the values so found be substituted in the equation,

$$X_1'^2 + Y_1'^2 + Z_1'^2 = \mu^2 R_1^2;$$

and if we find $\mu <$ or $= \mu_1$, the conditions of equilibrium are satisfied, and the particle remains at rest. If $\mu_1 - \mu$ be finite, the particle will continue at rest for a time which cannot be indefinitely small.

If $\mu = \mu_1$, the particle will remain at rest or commence to move again, according as $\frac{d\mu}{dt}$ is negative or positive. If this coefficient vanish, the result will depend on the sign of the lowest differential coefficient which does not vanish.

In any of these cases it is evident that there will be a suffi-

cient number of equations to determine all the unknown quantities of the problem. And it appears, from the foregoing discussion (p. 101), that this will always be so, unless a group of particles, between whose co-ordinates there exists a geometrical equation, be at rest simultaneously. Suppose now that there be such a group consisting of p particles. Then, so long as any one of these particles continues to move, all the forces will be determinate. Moreover, at the moment when the p^{th} particle comes to rest, all these forces are still determinate. But if we adhere to the abstractions of Rational Mechanics, there will be thenceforward nothing to determine the intensity of the geometrical force corresponding to the equation which connects the co-ordinates of these particles while they continue at rest. Thus every group of quiescent particles between which there exists a geometrical equation, will introduce into the solution one indeterminate quantity.

If the system start from a state of rest, the solution will, in general, be indeterminate for another reason—namely, the impossibility of determining which of the particles remain at rest, and which of them commence to move. But, on account of the importance of the problem of initial motion, we shall consider it separately.

III.—*Initial Movement of a System of Particles.*

9. A system of material particles connected by given geometrical relations is, in general, geometrically capable of an infinite number of systems of movements, only those movements which are inconsistent with the geometrical relations being excluded. But we know that only one system of movements is *dynamically* possible; that is to say, that if the dynamical conditions of the system be also fully given, the motion is determinate and unique. Now, the dynamical conditions *are* fully given, if we know the positions of the particles, the forces which act upon them, and their velocities at any given time. Hence, if the forces which act upon the system be perfectly determinate, the *initial* motion of the system—that is to say, the motion which succeeds to an original state of rest—is also determinate and unique.

But it is plain that this conclusion does not necessarily hold if any of the forces which act upon the system be in their nature *in*determinate. If, all the other conditions remaining the same, any of the acting forces be capable of assuming different values, there would be, in general, for every distinct system of values assumed by these forces a distinct system of initial movements. Now, with regard to the forces commonly called *external*, these are, in general, perfectly determinate, depending entirely upon the positions of the particles of which the system is composed. But it is otherwise, as we have seen, with the geometrical forces. The intensities of these forces are not given functions of the co-ordinates, but distinct unknown quantities, to be determined, if they can be determined, by the equations of motion themselves. So also with the forces of friction, which depend, as we have seen, upon the other forces of the system, and, therefore, upon the geometrical forces. Hence, it is plain that a system acted on by given external forces, and having a given initial position, *may* be capable of more than one kind of initial motion. It becomes then an important question to decide in what cases this indeterminateness really exists, and in what cases such a system is capable of but one initial motion.

10. It is easily shown by the principles of ordinary dynamics that, if the supporting surfaces be smooth, the initial motion and all subsequent motions are perfectly determinate. For, let n be the number of the particles of which the system is composed, and m the number of the geometrical conditions by which they are restricted, the equations of the supporting surfaces being included in these relations. The unknown quantities of the problem are then—1. The co-ordinates of the several particles, in number $3n$. 2. The intensities of the geometrical forces, in number m. For the determination of these quantities we have the $3n$ differential equations of motion, and the m geometrical equations. These equations, which are in number equal to the unknown quantities, will, when integrated, determine these latter in terms of the time and $6n$ constants, the constants being themselves determined from the $3n$ initial values of the co-ordinates and the $3n$ initial values of the velocities. If, as we here suppose to be the case, the system start from rest, the $3n$

initial velocities are severally zero, but the solution is not the less determinate.

11. Let us now consider the case in which the supporting surfaces are rough. Here, as we have seen (p. 99), the equations of the problem enable us to determine completely the condition of the moving particles, if we know that they *are* moving. But as these equations do not give us any means of deciding *which* of the particles are at rest and *which* are in motion, the initial motion of the system, in the sense in which that term is here used, remains still indeterminate. We shall now proceed to show, in the first place, that part of this ambiguity may be removed by the exclusion of systems of movements, which, though geometrically possible, are dynamically impossible.

For this purpose we shall, in the first place, investigate the following proposition :—

PROP. II.

12. A number, n, of material particles, resting each upon a rough surface, are connected by known relations, and acted on by given forces. If k of these particles commence to move, the impressed velocities being either nothing or indefinitely small, the remaining $n-k$ particles continuing at rest, to determine the values of the geometrical forces, and of the forces of resistance for all the particles of the system.

Recurring to the notation of p. 98, we have, for the k moving particles, equations of motion of the form,

$$m_1 \frac{d^2x_1}{dt^2} = X_1 + X'_1 + \alpha_1 R_1 + \lambda_1 \frac{dL_1}{dx_1} + \lambda_2 \frac{dL_2}{dx_1} + \&c.$$

$$m_1 \frac{d^2y_1}{dt^2} = Y_1 + Y'_1 + \beta_1 R_1 + \lambda_1 \frac{dL_1}{dy_1} + \lambda_2 \frac{dL_2}{dy_1} + \&c. \qquad (19)$$

$$m_1 \frac{d^2z_1}{dt^2} = Z_1 + Z'_1 + \gamma_1 R_1 + \lambda_1 \frac{dL_1}{dz_1} + \lambda_2 \frac{dL_2}{dz_1} + \&c.$$

and for the $n-k$ particles which are at rest, equations of equilibrium of the form,

$$o = X_{k+1} + X'_{k+1} + \alpha_{k+1} R_{k+1} + \lambda_1 \frac{dL_1}{dx_{k+1}} + \lambda_2 \frac{dL_2}{dx_{k+1}} + \&c.$$

$$o = Y_{k+1} + Y'_{k+1} + \beta_{k+1} R_{k+1} + \lambda_1 \frac{dL_1}{dy_{k+1}} + \lambda_2 \frac{dL_2}{dy_{k+1}} + \&c. \qquad (20)$$

$$o = Z_{k+1} + Z'_{k+1} + \gamma_{k+1} R_{k+1} + \lambda_1 \frac{dL_1}{dz_{k+1}} + \lambda_2 \frac{dL_2}{dz_{k+1}} + \&c.$$

Now, if we twice differentiate the first k equations of the supporting surfaces with regard to the time, and after the second differentiation, reject the terms containing the squares and products of

$$\frac{dx_1}{dt},\ \frac{dy_1}{dt},\ \frac{dz_1}{dt},\ \frac{dx_2}{dt},\ \&c.$$

since the initial velocities are zero, or indefinitely small, we shall have k equations of the form

$$\frac{du_1}{dx_1}\frac{d^2x_1}{dt^2} + \frac{du_1}{dy_1}\frac{d^2y_1}{dt^2} + \frac{du_1}{dz_1}\frac{d^2z_1}{dt^2} = o,\ \&c.$$

Substituting for the second differential coefficients from the equations of motion, we have the k equations,

$$o = \frac{du_1}{dx_1}(X_1 + X'_1 + \alpha_1 R_1 + \&c.) + \frac{du_1}{dy_1}(Y_1 + Y'_1 + \beta_1 R_1 + \&c.) \qquad (21)$$
$$+ \frac{du_1}{dz_1}(Z_1 + Z'_1 + \gamma_1 R_1 + \&c.)$$

Again, we know that when a particle is in motion the force of friction is directly opposed to the motion of the particle. Now, if the particle start without any impressed velocity, the initial motion must be in the direction of the tangential component of the initial force. Hence, in this case, if we compound into a single force all the acting forces, external and geometrical, and resolve this resultant perpendicular and parallel to the tangent plane, the force of friction will be directly opposed to the latter component. (It is obviously indifferent whether in thus compounding the acting forces, we include or exclude the force

of friction itself, or the normal resistance.) Since, then, the line of direction of the force of friction lies in the tangent plane, we must have

$$a_1 X'_1 + \beta_1 Y'_1 + \gamma_1 Z'_1 = 0. \tag{22}$$

Moreover, it is plain from what has been said, that the line of direction of this force lies in the plane containing the normal to the surface and the total acting force; hence, if we denote by A, B, C, the rectangular components of this latter force, it is easily seen that

$$(C\beta_1 - B\gamma_1) X'_1 + (A\gamma_1 - Ca_1) Y'_1 + (Ba_1 - A\beta_1) Z'_1 = 0;$$

or, replacing A, B, C by their values, as given in (19), omitting the normal resistance and the force of friction (which would evidently disappear from the equation of themselves), we have

$$\begin{aligned} &\left\{\left(Z_1 + \lambda_1 \frac{dL_1}{dz_1} + \&c.\right)\beta_1 - \left(Y_1 + \lambda_1 \frac{dL_1}{dy_1} \quad \&c.\right)\gamma_1\right\} X'_1 \\ + &\left\{\left(X_1 + \lambda_1 \frac{dL_1}{dy_1} + \&c.\right)\gamma_1 - \left(Z_1 + \lambda_1 \frac{dL_1}{dz_1} + \&c.\right)a_1\right\} Y'_1 \quad (23) \\ + &\left\{\left(Y_1 + \lambda_1 \frac{dL_1}{dy_1} + \&c.\right)a_1 - \left(X_1 + \lambda_1 \frac{dL_1}{dx_1} + \&c.\right)\beta_1\right\} Z'_1 = 0. \end{aligned}$$

If the initial velocity be not zero, its direction will be given, and, therefore, the direction of the force of friction in the tangent plane. We shall, therefore, have, in addition to the equation (22), a second equation of the form

$$PX'_1 + QY'_1 + RZ'_1 = 0. \tag{24}$$

P, Q, R being known coefficients.

Again, since the particle is in motion, the force of friction has its greatest value—namely, $\mu_1 R$. We have then a third equation, *sc.*

$$X_1'^2 + Y_1'^2 + Z_1'^2 = \mu_1^2 R_1^2. \tag{25}$$

It is plain, therefore, that for every moving particle we have the four equations (21), (22), (25), and (23) or (24). For each quiescent particle we have the three equations of equilibrium,

(20). Moreover, the equation (22) expressing the fact that the force of friction is in the tangent plane, is true for every particle, whether in motion or at rest. In every case, therefore, whether of rest or motion, the number of equations is equal to the number of the unknown quantities X'_1, Y'_1, Z'_1, R_1, depending upon the forces of resistance.

With regard to the geometrical forces, the number of unknown quantities which are introduced by them is, as we know, the same as the number of the connecting equations. Let $L = 0$ be one of these equations. Then, if this equation be twice differentiated, rejecting small quantities as before, and substituting the values of the second differential coefficients, we obtain a linear equation between the unknown quantities of the problem. So also of all the connecting equations, provided, as in p. 99, that no one of these equations contain only the coordinates of quiescent particles. If there be no such equation, the number of equations is evidently the same as the number of unknown quantities, which are, therefore, perfectly determinate.

13. If now, on substituting the values of X'_1, Y'_1, Z'_1, R for any of the quiescent particles, in the equation

$$X'^2 + Y'^2 + Z'^2 = \mu'^2 R^2,$$

we find a value for μ' greater than the coefficient of friction at that point, it is evident that the supposed movement is impossible. All such movements, therefore, must be rejected.

Again, we must reject every system of movements for which the necessary effect of the forces acting on any one of the moving particles is to diminish its velocity. For, it is plain that such a force could not set the particle in motion. Now, since there is no impressed velocity, if the acting forces, external and geometrical, be resolved perpendicular and parallel to the tangent plane, the initial motion, if there be any, will be in the direction of the resultant of these latter components. Unless, then, this resultant exceed the force of friction, the supposed movement is impossible. Now, we have seen that for a moving particle all the forces are determinate. Let V be the resultant of the forces external and geometrical (excluding the force of friction) resolved along the tangent plane, and R the normal

reaction of the supporting surface. Then it is evident, from what has been said, that every system of initial movements must be rejected as dynamically impossible, which does not satisfy the condition,

$$V - \mu R > 0.$$

14. But, after every system of movements has been rejected except those which are, both geometrically and dynamically, possible, it will frequently be found that the question remains still indeterminate. We have seen, for example, that this will be, in general, the case, when there exists an equation of condition involving only the co-ordinates of quiescent particles. The cause of this indeterminateness has been fully considered before (pp. 19, 20, 89, 90).

Example.

15. Two material particles, m_1, m_2, rest upon a rough inclined plane at the points A, B, and are connected by a thin string passing through a small smooth ring fixed to a point, O, in the plane. If the inclination of the plane be gradually increased till equilibrium be broken, to determine the nature of the initial movement.

It is evident that the initial movement of the system may be one of two kinds. 1. One of the particles may move, the other remaining fixed. 2. Both particles may move simultaneously. We shall consider these species of movements successively.

(1.) If the particle m_1 move, while m_2 remains fixed, the motion will necessarily be in a circle described round O. The force of friction at A is, therefore, perpendicular to OA, and has its maximum value.

Let the plane itself be taken for the plane of xy, O being the origin, and the axis of x parallel to the horizon. Let a be the inclination of the plane at the commencement of the motion, and let the angles AOY, BOY be represented by β_1, β_2, respectively, the positive direction of Y being downwards. Then if $x_1\, y_1$ be the co-ordinates of m_1, we have the equations

$$m_1 \frac{d^2x_1}{dt^2} = - T \sin \beta_1 + \mu m_1 g \cos \alpha \cos \beta_1$$

$$m_1 \frac{d^2y_1}{dt^2} = - T \cos \beta_1 - \mu m_1 g \cos \alpha \sin \beta_1 + m_1 g \sin \alpha. \quad (26)$$

T being the geometric force (or the force of tension).

Again, for the quiescent particle, m_2, we have

$$0 = - T \sin \beta_2 + X'_2$$

$$0 = - T \cos \beta_2 + Y'_2 + m_2 g \sin \alpha, \quad (27)$$

where X'_2, Y'_2 are the components of the effective force of friction at B. Now, since the particle m_1 moves in a circle round O, we have

$$x_1^2 + y_1^2 = \text{const.}$$

Differentiating this equation according to the method of p. 106, we have

$$x_1 \frac{d^2x_1}{dt^2} + y_1 \frac{d^2y_1}{dt^2} = 0, \text{ or, } \sin \beta_1 \frac{d^2x_1}{dt^2} + \cos \beta_1 \frac{d^2y_1}{dt^2} = 0.$$

Substituting for the differential coefficients from (26), we find

$$T = m_1 g \sin \alpha \cos \beta_1. \quad (28)$$

Hence, from equations (27),

$$X'_2 = m_1 g \sin \alpha \cos \beta_1 \sin \beta_2, Y'_2 = m_1 g \sin \alpha \cos \beta_1 \cos \beta_2 - m_2 g \sin \alpha. \quad (29)$$

We have, therefore,

$$X_2'^2 + Y_2'^2 = g^2 \sin^2 \alpha \, (m_1^2 \cos^2 \beta_1 - 2 m_1 m_2 \cos \beta_1 \cos \beta_2 + m_2^2).$$

Now, since the resultant of X_2', Y_2' must not exceed the maximum force of friction, $\mu m_2 g \cos \alpha$, we have

$$\mu^2 m_2^2 \cot^2 \alpha >, \text{ or } = m_1^2 \cos^2 \beta_1 - 2 m_1 m_2 \cos \beta_1 \cos \beta_2 + m_2^2. \quad (30)$$

Again, since the resultant of the tension and the force of gravity acting on the moving particle m_1 must exceed the maximum force of friction, we must have

$$T^2 \sin^2 \beta_1 + (m_1 g \sin \alpha - T \cos \beta_1)^2 > \mu^2 m_1^2 g^2 \cos^2 \alpha;$$

or, substituting for T from (28),

$$\mu^2 \cot^2 a < \sin^2 \beta_1.$$

Hence, and from (30), we have, putting $m_1 = nm_2$,

$$\sin^2 \beta_1 > 1 - 2n \cos \beta_1 \cos \beta_2 + n^2 \cos^2 \beta_1,$$

or
$$(1 + n^2) \cos^2 \beta_1 - 2n \cos \beta_1 \cos \beta_2 < 0. \tag{31}$$

It is evident that $\cos \beta_1$ and $\cos \beta_2$ cannot both be negative, inasmuch as in that case both particles would be *above* the fixed ring, and would, therefore, necessarily move as independent particles, this movement commencing as soon as $\mu \cot a$ had fallen below the value 1. We learn, then, from (31) that neither $\cos \beta_1$ nor $\cos \beta_2$ can be negative; and, therefore, that both particles must be situated *below* the fixed ring.

Moreover, it appears from (31) that $\cos \beta_1$ must be less than $\cos \beta_2$. Hence, we infer that if either particle move while the other remains at rest, the moving particle must be that for which the connecting string is most oblique to the axis of y, and, therefore, most oblique to the vertical. If the condition (31) be satisfied, the particle m_1 may commence to move by itself as soon as the inclination of the plane exceeds the value given by the equation

$$\tan a = \mu \cos \beta_1.$$

If

$$\frac{\cos \beta_1}{\cos \beta_2} > \frac{2m_1 m_2}{m_1^2 + m_2^2} < \frac{m_1^2 + m_2^2}{2m_1 m_2}, \tag{32}$$

neither particle can move separately.

(2.) If both particles move simultaneously, we have for the movement of m_1 the equations

$$\left.\begin{aligned} m_1 \frac{d^2 x_1}{dt^2} &= - T \sin \beta_1 + X_1' \\ m_1 \frac{d^2 y_1}{dt^2} &= - T \cos \beta_1 + Y_1' + m_1 g \sin a. \end{aligned}\right\} \tag{33}$$

Similar equations hold for the movement of m_2. We have also,

from the geometrical condition of the inextensibility of the string,

$$\sqrt{x_1^2 + y_1^2} + \sqrt{x_2^2 + y_2^2} = \text{const}.$$

Differentiating this equation as before, we find

$$\sin\beta_1 \frac{d^2x_1}{dt^2} + \cos\beta_1 \frac{d^2y_1}{dt^2} + \sin\beta_2 \frac{d^2x_2}{dt^2} + \cos\beta_2 \frac{d^2y_2}{dt^2} = 0.$$

Substituting for the differential coefficients from the equations of motion, we find

$$T\left(\frac{1}{m_1} + \frac{1}{m_2}\right) = \frac{X_1'}{m_1}\sin\beta_1 + \frac{Y_1'}{m_1}\cos\beta_1 + \frac{X_2'}{m_2}\sin\beta_2 + \frac{Y_2'}{m_2}\cos\beta_2 + g\sin a(\cos\beta_1 + \cos\beta_2). \quad (34)$$

Again, since there is no impressed velocity, the line of direction of the initial motion, and, therefore, the line of direction of the force of friction, must coincide with the resultant of T and the resolved force of gravity. Hence, for the particle m_1,

$$\frac{Y_1'}{X_1'} = \cot\beta_1 - \frac{m_1 g\sin a}{T\sin\beta_1};$$

and since the force of friction has its maximum value,

$$Y_1'^2 + X_1'^2 = \mu^2 m_1^2 g^2 \cos^2 a.$$

Hence,

$$X_1' = \frac{\mu m_1 g T\sin\beta_1}{V_1}\cos a,\quad Y_1' = \frac{\mu m_1 g(T\cos\beta_1 - m_1 g\sin a)}{V_1}\cos a, \quad (35)$$

where

$$V_1^2 = T^2 - 2m_1 g T\sin a\cos\beta_1 + m_1^2 g^2 \sin^2 a.$$

Similar values are found for X_2', Y_2'. Substituting these values in (34), we obtain an equation for T—namely,

$$T\left\{\frac{1}{m_1} + \frac{1}{m_2} - \mu\left(\frac{1}{V_1} + \frac{1}{V_2}\right)g\cos a\right\} - g\sin a\left\{\cos\beta_1 + \cos\beta_2 - \mu g\cos a\left(\frac{m_1\cos\beta_1}{V_1} + \frac{m_2\cos\beta_2}{V_2}\right)\right\} = 0. \quad (36)$$

It is easy to see that this equation has a real positive root. For, if we make $T = 0$, the left-hand ember of the equation becomes

$$(\cos\beta_2 + \cos\beta_1)\ \mu \cos\alpha - \sin\alpha)g,$$

a negative quantity, since $\mu \cos\alpha < \sin\alpha$. If, on the other hand, we make $T = \infty$, it is evident that the left-hand member of the equation will be positive.

It is further necessary to the motion of the two particles that at each point the resultant of the tension and the force of gravity should exceed the maximum force of friction. We must have, therefore,

$$V_1 > \mu m_1 g \cos\alpha, \quad V_2 > \mu m_2 g \cos\alpha.$$

Now, the equation (36) may be put under the form

$$\left.\begin{aligned} &m_2 V_2 (T - m_1 g \sin\alpha\cos\beta_1)(V_1 - \mu m_1 g \cos\alpha) \\ &+ m_1 V_1 (T - m_2 g \sin\alpha\cos\beta_2)(V_2 - \mu m_2 g \cos\alpha) = 0. \end{aligned}\right\} \quad (37)$$

But, inasmuch as the first and third factors in each of these products are essentially positive, the remaining factors must have opposite signs. Hence, the value of T must lie within the limits $m_1 g \sin\alpha\cos\beta_1$, and $m_2 g \sin\alpha\cos\beta_2$.

To determine the least inclination of the plane for which movement of both particles is possible.

Considering T as a function of α, determined by the equation (37), we have to solve the following problem :—

To find the smallest value of α which will satisfy the conditions

$$V_1 - \mu m_1 g \cos\alpha > 0, \quad V_2 - \mu m_2 g \cos\alpha > 0.$$

Here it is plain that the limiting value of α is found by equating one of these quantities to zero. Let

$$V_1 - \mu m_1 g \cos\alpha = 0. \qquad (38)$$

Then, from equation (37) we have either

$$V_2 - \mu m_2 g \cos\alpha = 0, \text{ or } T - m_2 g \sin\alpha\cos\beta_2 = 0. \qquad (39)$$

(a). The former of these alternatives combined with (38) gives

$$T^2 - 2m_1gT\sin\alpha\cos\beta_1 + m_1^2g^2\sin^2\alpha = \mu^2m_1^2g^2\cos^2\alpha$$
$$T^2 - 2m_2gT\sin\alpha\cos\beta_2 + m_2^2g^2\sin^2\alpha = \mu^2m_2^2g^2\cos^2\alpha. \tag{40}$$

Subtracting these equations one from the other, we have

$$2T(m_1\cos\beta_1 - m_2\cos\beta_2) = (m_1^2 - m_2^2)(1 - \mu^2\cot^2\alpha)\,g\sin\alpha. \tag{41}$$

Let $m_1 > m_2$. Then, since the inclination of the plane is greater (or, at least, not less) than the angle of friction, $1 - \mu^2\cot^2\alpha$ is positive. Hence

$$m_1\cos\beta_1 > m_2\cos\beta_2.$$

Again, multiplying equations (40) by m_2^2, m_1^2, respectively, and subtracting, we have

$$(m_1^2 - m_2^2)\,T = 2m_1m_2(m_1\cos\beta_2 - m_2\cos\beta_1)\,g\sin\alpha. \tag{42}$$

Hence, evidently,

$$m_1\cos\beta_2 > m_2\cos\beta_1.$$

The suppositions

$$V_1 - \mu m_1g\cos\alpha = 0,\quad V_2 - \mu m_2g\cos\alpha = 0,$$

are, therefore, impossible, unless

$$\frac{\cos\beta_1}{\cos\beta_2} > \frac{m_2}{m_1} < \frac{m_1}{m_2}. \tag{43}$$

If the ratio of the cosines lie within these limits, the value of α is found from (41) and (42). We have then

$$1 - \mu^2\cot^2\alpha = \frac{4m_1m_2}{(m_1^2 - m_2^2)^2}(m_1\cos\beta_1 - m_2\cos\beta_2)(m_1\cos\beta_2 - m_2\cos\beta_1)$$

It is easily shown that the value of α obtained from this equation is less than that obtained for the case in which one particle moves separately. If, then, the ratio of the cosines lie within the limits (43), the inclination at which the movement of both particles is possible will be obtained sooner than the inclination at which one particle may move separately. Moreover, if the ratio of the cosines lie within the limits (32), in which case separate motion is impossible (it will, as is easily seen) lie within the limits (43), in which case joint movement of the kind here discussed is possible.

(*b*). With regard to the second alternative, $T - m_2 g \sin a \cos \beta_2 = 0$, it is easily seen to be, in general, inadmissible. For we have, as in p. 112,

$$\frac{Y_2'}{X_2'} = \cot \beta_2 - \frac{m_2 g \sin a}{T \sin \beta_2} = - \sin \beta_2.$$

Hence, it is evident that the direction of the force of friction, and, therefore, of the initial motion of m_2 is perpendicular to the string. But if this be true for m_1, it must also be true for m_1. Hence we can infer, as in p. 110, that $T = m_1 g \sin a \cos \beta_1$. Equating these values of T, we find

$$m_1 \cos \beta_1 = m_2 \cos \beta_2.$$

Unless, then, this condition is satisfied by the positions of the particles, the second alternative is inadmissible

IV.—*Motion of a Particle upon a Surface which has itself a given Motion.*

It is essential to the development of friction that the external force which acts upon the particle should tend to produce movement of the particle *on* the supporting surface. No force, therefore, is capable of developing friction, which would not produce motion on the supporting surface, if that surface were smooth.

So also with regard to the effect of motion. Only the *relative* movement of the particle and the supporting surface has any effect upon the friction developed; and the connexion between this force and relative motion, when the supporting surface is moveable, is the same as its connexion with absolute motion when the supporting surface is fixed. The law in either case may be expressed by saying that the force of friction is directed in the tangent to the path of the particle *on the surface,* and that its intensity is proportional to the pressure.

PROP. III.

A material particle moves under the influence of a given force upon a surface which is constrained to move according to a given law. To determine the equations of motion of the particle.

Let x', y', z' be the co-ordinates of a point on the moveable surface. Then, since the motion of the surface is given, we have evidently,

$$\frac{dx'}{dt} = \phi\,(x', y', z', t),\quad \frac{dy'}{dt} = \psi(x', y', z', t),\quad \frac{dz'}{dt} = \chi\,(x', y', z', t), \quad (44)$$

ϕ, ψ, χ being known functions.

Let x, y, z be the co-ordinates of the moving particle. Then, if α, β, γ be the angles which the tangent to the path of the particle on the surface make with the axis of the co-ordinates, it is known that

$$\cos\alpha = V\left(\frac{dx}{dt} - \frac{dx'}{dt}\right),\quad \cos\beta = V\left(\frac{dy}{dt} - \frac{dy'}{dt}\right),$$
$$\cos\gamma = V\left(\frac{dz}{dt} - \frac{dz'}{dt}\right), \quad (45)$$

where

$$V^{-2} = \left(\frac{dx}{dt} - \frac{dx'}{dt}\right)^2 + \left(\frac{dy}{dt} - \frac{dy'}{dt}\right)^2 + \left(\frac{dz}{dt} - \frac{dz'}{dt}\right)^2.$$

Let N be the normal resistance, and a, β, γ the cosines of the angles which it makes with the axis. Let, also, X, Y, Z be the components of the acting force. Then the equations of motion are,

$$\frac{d^2x}{dt^2} = X + aN - \mu NV\left(\frac{dx}{dt} - \frac{dx'}{dt}\right),$$
$$\frac{d^2y}{dt^2} = Y + \&c.,\quad \frac{d^2z}{dt^2} = Z + \&c. \quad (46)$$

where the values of the differential coefficients of x', y', z' are to be substituted from equations (44).

Since, moreover, the co-ordinates x, y, z always denote the same point as the co-ordinates x', y', z', we may put $x' = x$, $y' = y$, $z' = z$ in the functions ϕ, ψ, χ. We have thus three equations between the unknown quantities x, y, z, N. The fourth equation is that of the supporting surface. Let $F(x_1, y_1, z_1) = 0$, be the equation of the surface, referred to axes fixed with regard to the surface. Then the equation of the same surface referred to the axes fixed in space, will be

$$F(ax + by + cz + e,\ a'x + b'y + c'z + e',\ a''x + b''y + c''z + e'') = 0,$$

$a\ b\ c\ e$, a', &c. being given functions of t.

Example 1.

A heavy particle is projected along a rough horizontal plane, which moves uniformly parallel to itself without rotation. To determine the movement of the particle.

Since the only force acting is directly opposed to the movement of the particle on the plane, it is evident that the relative path will be a right line; and, since the intensity of this force is constant, the velocity in this path will be uniformly diminished.

The absolute motion of the particle in space is found by combining the uniformly retarded motion in this relative path with the uniform motion by which the path is carried parallel to itself. The path of the particle in space is evidently a parabola.

Example 2.

A rough horizontal plane revolves round a vertical axis with a constant angular velocity. Required to determine the motion of a particle placed without initial velocity on the plane at any point other than the centre of rotation.

Let ω be the angular velocity of the plane, $x\,y$ the co-ordinates of any point on it, referred to fixed axes in the plane, the centre of rotation being the origin. Then the components of the velocity of the point $x\,y$, considered a point in the plane, are $\omega\,y$ parallel to x, and $-\omega\;x$ parallel to y. Hence, the components of the relative velocity of the material particle are

$$\frac{dx}{dt} - \omega y,\ \frac{dy}{dt} + \omega x.$$

Since, then, the direction of the force of friction is directly opposed to the relative velocity of the particle, the equations of motion are

$$\frac{d^2x}{dt^2} = -\frac{\mu g}{v}\left(\frac{dx}{dt} - \omega y\right),\quad \frac{d^2y}{dt^2} = -\frac{\mu g}{v}\left(\frac{dy}{dt} + \omega x\right),$$

where

$$v^2 = \left(\frac{dx}{dt} - \omega y\right)^2 + \left(\frac{dy}{dt} + \omega x\right)^2, \qquad (47)$$

the mass of the particle being supposed = 1.

If the co-ordinates be transformed, so that the axes, instead of being fixed in space, shall revolve with the plane, these equations become

$$\left.\begin{aligned} \frac{d^2x}{dt^2} + 2\omega\frac{dy}{dt} - \omega^2 x &= -\mu g\frac{dx}{ds} \\ \frac{d^2y}{dt^2} - 2\omega\frac{dx}{dt} - \omega^2 y &= -\mu g\frac{dy}{ds}, \end{aligned}\right\} \tag{48}$$

ds being the element of the curve described by the particle on the plane. If the independent variable be changed from t to s, as in p. 91, we find

$$\left.\begin{aligned} v^2\frac{d^2x}{ds^2} + \left(v\frac{dv}{ds} + \mu g\right)\frac{dx}{ds} + 2\omega v\frac{dy}{ds} - \omega^2 x &= 0 \\ v^2\frac{d^2y}{ds^2} + \left(v\frac{dv}{ds} + \mu g\right)\frac{dy}{ds} - 2\omega v\frac{dx}{ds} - \omega^2 y &= 0. \end{aligned}\right\} \tag{49}$$

Multiplying these equations by dx, dy, respectively, and adding, we have

$$vdv \times \mu gds - \omega^2(xdx + ydy) = 0.$$

Whence, by integration,

$$v^2 + 2\mu gs - \omega^2(x^2 + y^2) = \text{const.} \tag{50}$$

But, since the initial value of the absolute velocity is zero, the initial value of v^2 is $\omega^2(x_0^2 + y_0^2)$, x_0, y_0 being the initial values of $x\, y$. Hence, evidently, the arbitrary constant in (50) vanishes.

Again, multiplying equations (49) by dy, dx, respectively, and subtracting, we find, without difficulty,

$$v^2 + 2\omega v\rho - \omega^2 p\rho = 0. \tag{51}$$

Where ρ is the radius of curvature, and p the perpendicular from the origin on the tangent.

Eliminating v between (50), and (51), we obtain the equation of the trajectory in a differential form, which does not seem to admit of integration.

Prop. IV.

If a material particle, acted on by given forces, be placed on a rough surface, which is itself in motion, to determine the conditions requisite in order that the particle may be at rest on the surface, the motion of the surface being supposed to be given.

Let x, y, z be the co-ordinates of the particle referred to fixed axes, X, Y, Z, the acting forces, R the reaction of the rough surface, and α, β, γ, the cosines of the angles which this reaction makes with the axes. Then the equations of motion of the particle in space are

$$\frac{d^2x}{dt^2} = X + \alpha R$$

$$\frac{d^2y}{dt^2} = Y + \beta R \qquad (52)$$

$$\frac{d^2z}{dt^2} = Z + \gamma R.$$

If, now, the particle be at rest on the surface, the motion of the particle in space must be the same as that of the point at which it was originally placed. But, since the motion of the surface is given, we have for the point $x\,y\,z$, considered as a point on the surface,

$$x = \phi(t),\ y = \psi(t),\ z = \chi(t),$$

ϕ, ψ, χ being given functions. Hence

$$\phi''(t) = X + \alpha R$$

$$\psi''(t) = Y + \beta R \qquad (53)$$

$$\chi''(t) = Z + \gamma R.$$

Let α', β', γ', be the cosines of the angles which the normal to the surface makes with the fixed axes. These cosines are known functions of the time. Then it is necessary and sufficient for equilibrium that

$$\alpha\alpha' + \beta\beta' + \gamma\gamma' > \text{or} = \cos\epsilon.$$

Substituting for α, β, γ from (53), we have

$$\frac{\alpha'(\phi''t - X) + \beta'(\psi''t - Y) + \gamma'(\chi''t - Z)}{\sqrt{(\phi''t - X)^2 + (\psi''t - Y)^2 + (\chi''t - Z)^2}} > \text{or} = \cos\epsilon. \qquad (54)$$

Example 4.

A material particle is placed on a rough horizontal plane, which is made to revolve round a vertical axis with a velocity commencing from zero, and continually increasing. To determine the velocity at which the particle will commence to move on the plane.

So long as the particle continues at the same point on the plane, its co-ordinates are given by the equations,

$$x = a \sin u, \; y = a \cos u, \; z = 0,$$

where $u = \int \omega dt$. We have then in Prop. IV.,

$$\phi'' t = - a\omega^2 \sin u + a \cos u \frac{d\omega}{dt}$$

$$\psi'' t = - a\omega^2 \cos u - a \sin u \frac{d\omega}{dt}.$$

Also,

$$X = 0, \; Y = 0, \; Z = -g, \; \alpha' = 0, \; \beta' = 0, \; \gamma' = 1.$$

Substituting these values in the condition (54), we find, without difficulty,

$$\frac{\mu^2 g^2}{a^2} > \text{or} = \omega^2 + \frac{d\omega^2}{dt^2}.$$

Hence, if the velocity increases more rapidly than is indicated by the law $a\omega = \mu g t$ the particle, will commence to move on the plane simultaneously with the commencement of the rotation of the plane itself.

CHAPTER V.

MOTION OF A SOLID BODY.

I.—*Motion of a Solid Body on a Rough Plane.*

1. The problem which we propose to discuss here is in general so complex that it does not admit of a complete solution, except in some of the simplest cases. All that can be done in the case of the general problem is to state the physical principles on which the solution depends, and to deduce the general equations, from the integration of which it might be obtained. We shall, in the first place, investigate the problem in its generality, and then consider some particular cases which admit of a solution more or less complete.

Let the fixed plane be taken for the plane of xy. Let $x_1\, y_1\, z_1$ be the co-ordinates of the centre of gravity, and ξ, η the components of the velocity of the point of contact of the moving body with the fixed plane, both referred to the fixed axes.

Conceive the forces acting on the body (not including the resistance of the rough plane) to be reduced to a single force passing through the centre of gravity, and a single moment taken with respect to the same point. Let X_1, Y_1, Z_1 be the components of the force, and L_1, M_1, N_1 the components of the moment. Let also X', Y', Z' be the components of the resistance of the rough plane, all these components being referred to the fixed axes.

Consider now the system of principal axes which pass through the centre of gravity. Let α, β, γ, be the co-ordinates of the point of contact referred to these axes, and abc, $a'b'c'$, $a''b''c''$ the cosines of the angles which they make with the fixed axes xyz.

Let p, q, r be the components of the angular velocity of the body referred to the principal axes, and θ, ϕ, ψ the ordinary polar angles which determine the position of these axes with regard to the fixed axes. Then, taking the mass of the body $= 1$, we have the following equations:—

(1.) For the motion of the centre of gravity,

$$\frac{d^2x_1}{dt^2} = X_1 + X', \quad \frac{d^2y_1}{dt^2} = Y_1 + Y', \quad \frac{d^2z_1}{dt^2} = Z_1 + Z'. \qquad (1)$$

(2.) For the rotation round the centre of gravity, A, B, C being the principal moments of inertia,

$$\begin{aligned} Adp + (C - B)\, qrdt &= (aL_1 + bM_1 + cN_1 + \beta Z - \gamma Y)\, dt \\ Bdq + (A - C)\, prdt &= (a'L_1 + b'M_1 + c'N_1 + \gamma X - \alpha Z)\, dt \qquad (2) \\ Cdr + (B - A)\, pqdt &= (a''L_1 + b''M_1 + c''N_1 + \alpha Y - \beta X)\, dt, \end{aligned}$$

where

$$X = aX' + a'Y' + a''Z', \quad Y = bX' + \&c., \quad Z = cX' + \&c.$$

(3.) We have, besides, the well-known equations,

$$\begin{aligned} pdt &= \sin\phi \sin\theta d\psi - \cos\phi d\theta \\ qdt &= \cos\phi \sin\theta d\psi + \sin\phi d\theta \qquad (3) \\ rdt &= d\phi - \cos\theta d\psi. \end{aligned}$$

Then, since abc, $a'b'c'$, $a''b''c''$ are all expressed in terms of θ, ϕ, ψ, these equations contain fifteen unknown quantities—namely, $x_1 y_1 z_1$, pqr, $\theta\phi\psi$, $\alpha\beta\gamma$, $X'Y'Z'$. We require, therefore, six equations in addition to those given above. These are obtained as follows:—

Let $u = f(x, y, z) = 0$ be the equation of the moving body referred to the principal axes. We have then,

$$f(\alpha, \beta, \gamma) = 0. \qquad (4)$$

Moreover, since the tangent plane at the point α, β, γ, is the plane of xy, we have

$$a\frac{du}{d\alpha} + a'\frac{du}{d\beta} + a''\frac{du}{d\gamma} = 0, \quad b\frac{du}{d\alpha} + b'\frac{du}{d\beta} + b''\frac{du}{d\gamma} = 0. \qquad (5)$$

Also, since z_1 is the perpendicular from the origin of the moveable axes upon the tangent plane at the point $\alpha\beta\gamma$, we have

$$z_1\sqrt{\frac{du^2}{d\alpha^2} + \frac{du^2}{d\beta^2} + \frac{du^2}{d\gamma^2}} = \alpha\frac{du}{d\alpha} + \beta\frac{du}{d\beta} + \gamma\frac{du}{d\gamma}. \qquad (6)$$

We have thus four of the six equations which are required.

The remaining two will differ according to the nature of the motion of the solid body.

If there be *slipping* at the point of contact—that is to say, if the point of the body which is in contact with the supporting plane have a finite velocity parallel to the plane—the force of friction will have its greatest value, and its direction will be immediately opposed to that of the resolved velocity.

Now, the velocity of the point of contact resolved parallel to the axes of x and y, respectively, will be

$$\begin{aligned}\xi &= \frac{dx_1}{dt} + a\,(q\gamma - r\beta) + a'\,(r\alpha - p\gamma) + a''\,(p\beta - q\alpha) \\ \eta &= \frac{dy_1}{dt} + b\,(q\gamma - r\beta) + b'\,(r\alpha - p\gamma) + b''\,(p\beta - q\alpha),\end{aligned} \tag{7}$$

and the two equations sought for are

$$X'\eta - Y'\xi = 0, \quad X'^2 + Y'^2 = \mu^2 Z'^2. \tag{8}$$

If, on the other hand, there be no slipping at the point of contact—that is to say, if the point of the body which is in contact with the supporting plane have no finite velocity parallel to that plane—the required equations are evidently

$$\xi = 0, \quad \eta = 0. \tag{9}$$

In this case the motion is a pure rotation round an axis passing through the point of contact.

2. The principle to be employed in deciding between these two alternatives is perfectly analogous to a principle before enunciated in Statics. As in the statical problem, the force of friction will always, if possible, reduce the particle which is in contact with the rough surface to rest; so in the dynamical problem, the force of friction will, if possible, determine such a motion of the moving body, that the point of the body which is in contact with the supporting surface shall have no finite velocity parallel to the surface. The second alternative is, therefore, always true, if it be possible. The rule to be adopted is, therefore, as follows:—

Solve the problem, adopting the second alternative. Having

obtained the values of X', Y', Z', substitute them in the equation

$$X'^2 + Y'^2 = \mu'^2 Z'^2, \tag{10}$$

and thus determine the value of μ'. If, then, we find μ' not to be greater than μ, the coefficient of friction, the solution is possible, and therefore true. If the value found for μ' be greater than μ, the solution is impossible, and we must re-solve the problem, adopting the first alternative.

It is evident that the nature of the motion may change from one of these species to the other. The times of such changes are determined as follows :—

Suppose that, having solved the problem on the second supposition, we find $\mu'^2 = f(t)$, where $f(0) < \mu$. Then, if for any time t_1 we find $f(t_1) = \mu^2$ (unless this be a maximum value), the motion passes at this instant from the second to the first kind, and the problem must be solved anew on the first supposition. Suppose that we obtain from this solution

$$\xi^2 + \eta^2 = F(t). \tag{11}$$

Then if for any value t_2 we find $F(t_2) = 0$ (unless this be a minimum value), the motion passes again to the second kind. Thus any number of changes may be determined.

Example.

3. A homogeneous sphere is projected with given velocities of translation and rotation along a rough horizontal plane. To determine the nature of the movement and the moment at which it becomes one of pure rolling.

Taking the given plane as the plane of xy, we have for the motion of the centre of gravity the equations

$$\frac{d^2x}{dt^2} = X, \quad \frac{d^2y}{dt^2} = Y, \tag{12}$$

where X, Y are the components of the force of friction. Also if u, v, w be the angular velocities round the axes of x, y, z, re-

spectively, k the radius of gyration round the centre, and r the radius of the sphere,

$$k^2 \frac{du}{dt} = Yr, \quad k^2 \frac{dv}{dt} = -Xr, \quad \frac{dw}{dt} = 0. \tag{13}$$

Let ξ, η, be the components of the velocity of the point of contact parallel to x and y, respectively. Then

$$\xi = \frac{dx}{dt} - rv, \quad \eta = \frac{dy}{dt} + ru. \tag{14}$$

Then we obtain from (12) and (13), putting $k^2 + r^2 = n^2k^2$,

$$\frac{d\xi}{dt} = n^2 X, \quad \frac{d\eta}{dt} = n^2 Y. \tag{15}$$

But, since the motion will, in general, be a mixture of slipping and rolling,

$$X\eta - Y\xi_1 = 0, \text{ and } X^2 + Y^2 = \mu^2 g^2.$$

Hence

$$X = -\mu g \frac{\xi}{\sqrt{\xi^2 + \eta^2}}, \quad Y = -\mu g \frac{\eta}{\sqrt{\xi^2 + \eta^2}}. \tag{16}$$

Substituting these values in the equations (15), we obtain readily

$$\xi d\xi + \eta d\eta = -\mu g dt \sqrt{\xi^2 + \eta^2}.$$

Whence, by integration,

$$\xi^2 + \eta^2 = (c - \mu g t^2). \tag{17}$$

Let v' be the initial velocity of the point of contact. Then, evidently, $c = v'^2$, and the moment at which pure rolling commences is given by the equation

$$\mu g t_1 = v'^2.$$

It is easily seen that this motion will continue indefinitely. For, at the moment when pure rolling commences, we have $\xi = 0$, $\eta = 0$; whence $X = 0$, $Y = 0$; and, therefore, from equations (15),

$$\frac{d\xi}{dt} = 0, \quad \frac{d\eta}{dt} = 0.$$

Again, differentiating equations (12), and equating to zero ξ, η, and their first differential coefficients, we find that the second differential coefficients also vanish, and so of all higher coefficients. We must have, therefore, generally, $\xi = 0$, $\eta = 0$, showing that pure rolling continues indefinitely.

Multiplying the equations (15) by η, ξ, respectively, and subtracting, we find

$$\eta d\xi - \xi d\eta = 0;$$

whence, by integration,

$$\eta = m\xi = \xi \tan a \text{ (suppose).}$$

We have then,

$$X = - \mu g \cos a, \quad Y = - \mu g \sin a.$$

Substituting these values in (12), and integrating, we have

$$x = at - \frac{1}{2} \mu g t^2 \cos a, \quad y = bt - \frac{1}{2} \mu g t^2 \sin a,$$

a, b being arbitrary constants, and the origin being taken at the initial position of the point of contact. Eliminating t between these two equations, we find

$$(ay - bx)(b \cos a - a \sin a) = \frac{1}{2} \mu g (x \sin a - y \cos a)^2. \quad (18)$$

Hence, the path of the point of contact, and, therefore, of the centre of the sphere, is in general a parabola.

The parabola will become a right line if $a \sin a = b \cos a$. This condition gives, evidently, $a\eta_0 - b\xi_0 = 0$, denoting by η_0, ξ_0, the initial values of ξ, η. Substituting for ξ_0, η_0 their values, $a - rv_0$, $b + ru_0$, we find

$$au_0 + bv_0 = 0.$$

Hence, if the rotation originally communicated to the body be resolved into two—one round a horizontal, and one round a vertical axis—the motion of the centre will be a right line, if the

horizontal axis be at right angles to the direction of the initial motion of the centre.

If the initial motions of translation and rotation be communicated to the sphere by a single impulse—as in the case of a billiard ball—the motion of the centre will be necessarily rectilinear.

For, if we take the origin, as before, at the initial position of the point of contact, and the axis of x parallel to the direction of the impulse, we shall have $b = 0$, $u_0 = 0$, and, therefore, from (14), $\eta_0 = 0$; whence $a = 0$. The equation (18) becomes, therefore, $y = 0$, denoting that the motion of the centre is parallel to the axis of x.

4. We proceed now to consider specially the case in which the motion of the solid body is a pure rotation round an axis, fixed or variable. This kind of motion may be thus defined—

The motion of a solid body is a pure rotation, if it be possible at each instant to assign a line such that if a plane be drawn through this line and any point of the moving body, the motion of that point shall be at that instant perpendicular to the plane so drawn.

The condition of pure rotatory movement may be otherwise expressed, as follows:—

It is known that the most general movement which can be given to a solid body may be conceived to be made up of the two following—namely, 1, a motion of translation common to every point in the body; 2, a motion of rotation round a point, which may be chosen arbitrarily. Now, the motion will be a pure rotation when (and only when), this decomposition having been effected, the motion of translation is at right angles to the axis of rotation.

This may be readily shown from the expressions given by Lagrange for the displacements of the different parts of a solid body. These are of the form

$$\begin{aligned} \delta x &= A + Qz - Ry \\ \delta y &= B + Rx - Pz \\ \delta z &= C + Py - Qx, \end{aligned} \tag{19}$$

where A, B, C, are the components of the motion of translation, and P, Q, R, are proportional to the index cosines of the axis of rotation. Now, if the motion be a pure rotation, the movements of all the particles must be at any given instant parallel to the same plane. Hence it must be possible to determine the coefficients α, β, γ so as to satisfy the equation,

$$\alpha\delta x + \beta\delta y + \gamma\delta z = 0, \tag{20}$$

independently of x, y, z. Substituting the values of δx, δy, δz, we find thus,

$$A\alpha + B\beta + C\gamma = 0, \frac{P}{\alpha} = \frac{Q}{\beta} = \frac{R}{\gamma}. \tag{21}$$

From these equations it is easy to see that

$$AP + BQ + CR = 0, \tag{22}$$

which equation evidently expresses the proposition enunciated above.

We shall now proceed to give some general propositions connected with this kind of motion, which will be found useful in our subsequent investigations.

PROP. I.

5. If the motion of a solid body be at each instant a pure rotation, the equation of rotation round the instantaneous axis will be

$$\frac{d}{d\theta} I\omega^2 = 2L; \tag{23}$$

where $d\theta$ is the elementary angle described by the rotating body, I the moment of inertia round the instantaneous axis, ω the velocity of rotation, and L the statical moment of the acting forces round the instantaneous axis.

This is easily proved from the general equation of *vis viva*,

$$\Sigma m \frac{d}{dt} v^2 = 2\Sigma m \left(X\frac{dx}{dt} + Y\frac{dy}{dt} + Z\frac{dz}{dt} \right),$$

or for a solid body,

$$\int \frac{d.v^2}{dt}\, dm = 2 \int \left(X \frac{dx}{dt} + Y \frac{dy}{dt} + Z \frac{dz}{dt} \right) dm. \qquad (24)$$

For, since the motion is a pure rotation, we have for each element dm, p being the perpendicular from dm on the axis,

$$v = p\omega ;$$

also, if ϕ be the angle which a plane passing through the molecule, x, y, z, and any given line, makes with a fixed plane, it is easily seen that the statical moment, round this line, of a force acting on the molecule will be given by the equation,

$$L = X \frac{dx}{d\phi} + Y \frac{dy}{d\phi} + Z \frac{dz}{d\phi}.$$

Multiply the equation (24) by ω or $\frac{d\phi}{dt}$, which, being the same for every element, may be introduced indifferently without or within the sign of integration; and it is easily seen that the equation may be written

$$\frac{d}{d\phi}\, \omega^2 \int p^2 dm = 2L,$$

which is identical with (23).

PROP. II.

6. Let the movements of all the points of a solid body be supposed to be parallel to the same fixed plane. Let G be its centre of gravity, and let a plane be drawn through it parallel to the

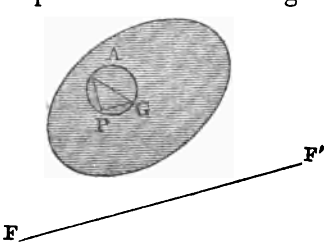

Fig. 19.

fixed plane. Let the plane of the paper represent the plane so drawn, and let A be the projection of the instantaneous axis,

which is, of course, perpendicular to this plane, and let a circle be described on the diameter, GA. Draw in the plane of the paper a line GP parallel to a fixed line, and through P a line perpendicular to the plane of the paper.

Then, if I be the moment of inertia, and L the statical moment of the acting forces with regard to this line, ω the angular velocity, and ϕ the actual angle of rotation, the equation obtained in the foregoing proposition for the instantaneous axis—namely,

$$\frac{d}{d\phi} I\omega^2 = 2L,$$

is true in this case also.

Since the movements of all sections parallel to the fixed plane are the same, it will be sufficient to consider motion in two dimensions. Let the axes of x and y be taken parallel, respectively, to the fixed directions, PA, GP. Assume $p = AP$ $q = GP$, and let k be the radius of gyration round an axis through G perpendicular to the plane of the paper. Then, since the motion is a pure rotation round the axis, whose projection is A, we have from the foregoing proposition, taking the mass of the body = 1,

$$\frac{d}{d\phi} (k^2 + p^2 + q^2)\, \omega^2 = 2L', \tag{25}$$

where L' is the statical moment with regard to the instantaneous axis. But if x, y, be the co-ordinates of any point in the body referred to axes through P, we have

$$L' = \int \{yX - (x - p)\ Y\}\, dm = L + p\int Ydm.$$

But from the motion of the centre of gravity, whose co-ordinates with regard to the fixed axis are $x_1\, y_1$, we have

$$\frac{d^2y_1}{dt^2} = \int Ydm.$$

Hence

$$L' = L + p\,\frac{d^2y_1}{dt^2}.$$

Now, since the motion of the centre of gravity is a pure rotation round an axis through A, we have

$$\frac{dy_1}{dt} = p\omega,$$

whence

$$\frac{d^2y_1}{dt^2} = \frac{d}{dt}p\omega = \omega\frac{d}{d\phi}p\omega;$$

and, therefore,

$$L' = L + p\omega\frac{d}{d\phi}p\omega = L + \frac{1}{2}\frac{d}{d\phi}p^2\omega^2.$$

Substituting this value in (25), and recollecting that $I = k^2 + q^2$, we have

$$\frac{d}{d\phi}I\omega^2 = 2L. \qquad (26)$$

7. The equations obtained in this and the foregoing propositions assume a particularly simple form when applied to the *initial* motion of a solid body. Performing the differentiation indicated in either of the equations (23) or (26), we have

$$\omega^2\frac{dI}{d\phi} + 2I\omega\frac{d\omega}{d\phi} = 2L,$$

or, since $d\phi = \omega dt$,

$$\omega\frac{dI}{dt} + 2I\frac{d\omega}{dt} = 2L.$$

But at the commencement of the motion, $\omega = 0$, hence the equation becomes

$$I\frac{d\omega}{dt} = I\frac{d^2\phi}{dt^2} = L, \qquad (27)$$

the same as the equation referred to the centre of gravity. Now, since ϕ is independent of the point to which the equation is referred, it is plain that, for all points situated on the circle defined as above, the ratio $L : I$ will be constant. This result, which we shall find useful afterwards, may be expressed as follows :—

If the initial motion of a solid body, acted on by any forces, be a pure rotation, and if a circle be described on the perpendicular from the centre of gravity upon the axis of rotation as diameter, the plane of the circle being perpendicular to the axis, the ratio of the statical moment to the moment of inertia is the same for every point on this circle.

Example.

8. A cylinder of any form descends a rough inclined plane, whose horizontal trace is parallel to the axis of the cylinder. If gravity be the sole force, determine the motion.

Let Fig. 20 represent a section of the cylinder and inclined plane by a plane perpendicular to the axis of the cylinder, which is necessarily a vertical plane. (It will evidently be sufficient to consider the motion of a single section). Let TP be perpendicular to the inclined plane, and GP perpendicular to TP, G being the centre of gravity of the section. Let GB be a fixed line in the section, and GQ perpendicular to the inclined plane. Then the rotation will be measured by the angle $BGQ = \theta$. Now, whether the motion of the cylinder be or be not pure rolling, the line TP must pass through the projection of the axis. For, if the motion be pure rolling, the point T is itself the projection of the axis, and if there be slipping at T, the motion at that point is parallel to the inclined plane, and, therefore, perpendicular to TP, which must, therefore, pass through the projection of the axis. Hence it is evident that, whatever point in TP be the projection of the axis, P must lie on the circle described as in the foregoing proposition. Moreover, GP is parallel to a fixed line. Hence, putting $GQ = TP = p$, and $TQ = GP = q$, and retaining the notation of the foregoing proposition, we have

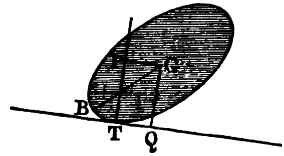

Fig. 20.

$$\frac{d}{d\theta}(k^2 + q^2)\,\omega^2 = 2(\mu N p + g q \cos \alpha). \qquad (28)$$

We have also for the centre of gravity G,

$$\frac{d}{d\theta} k^2\omega^2 = 2N(\mu p + q). \qquad (29)$$

Eliminating N between these equations, we find

$$\frac{d}{d\theta}(k^2 + q^2)\,\omega^2 = \frac{\mu p}{\mu p + q}\frac{d}{d\theta}\,k^2\omega^2 + 2gq\cos a,$$

or, performing the differentiations and reducing

$$\left(\frac{k^2}{\mu p - q} - q\right)\frac{d\omega^2}{d\theta^2} + 2\,\frac{dq}{d\theta}\,\omega^2 + 2g\cos a = 0. \qquad (30)$$

Now, the equation of the section of the cylinder being given, we have evidently,

$$p = f\theta,\ q = \frac{dp}{d\theta} = f'\theta.$$

Hence the foregoing equation is of the form

$$\frac{d\omega^2}{d\theta} + \omega^2\phi\theta = \psi\theta, \qquad (31)$$

a linear equation of the first order, integrable by quadratures, and giving

$$\omega^2 = F(\theta,c),$$

the constant being determined by means of the initial velocity. Replacing ω by its value, and integrating again,

$$t = \int\frac{d\theta}{\sqrt{F(\theta,c)}}, \qquad (32)$$

whence we have $\theta = \chi(t)$, determining the position of the cylinder at any moment. The angular velocity is, of course, found by the equation $\omega = \chi'(t)$.

The problem is, therefore, completely solved for the case where the coefficient of effective of friction has its maximum value, μ; or, in other words, for the case in which there is slipping as well as rolling.

If the motion be one of pure rolling, the coefficient of effective friction will be variable, but the equations (28) and (29) will still hold if μ be replaced by an unknown function of θ. Moreover,

since in this case the instantaneous axis always passes through T, the theorem demonstrated in Prop. I. gives the further equation,

$$\frac{d}{d\theta}(k^2 + p^2 + q^2)\,\omega^2 = 2g\,(p\sin\alpha + q\cos\alpha). \tag{33}$$

This equation admits of being integrated at once when the equation of the section is known, giving ω^2 in terms of θ, and, as before, θ in terms of t.

If, now, in the equation (30) we replace μ by an unknown function, Θ, and substitute the value of ω^2, obtained from (33), we shall have an equation to determine Θ in terms of θ; and, therefore, in terms of t. If this value (positive or negative) do not exceed $\pm\mu$, the motion is one of pure rolling. If the value of Θ lie outside these limits, the motion is slipping as well as rolling, and we must recur to the former solution.

In the preceding discussion the centre of gravity is supposed to be *in front of* the perpendicular to the inclined plane through T. All the equations are made applicable to the case in which the centre of gravity lies behind this line by simply changing the sign of q. They are made applicable to all cases if we replace q by its value $\frac{dp}{d\theta}$. For it is easily seen that this expression changes sign as the centre of gravity passes through the perpendicular to the inclined plane.

We now proceed to consider generally the case of a rolling body.

Prop. III.

9. If a solid body of any form roll without sliding upon a rough plane, the absolute accelerating force at any point of the body is equal to the relative force with regard to the point of the body which is in contact with the plane.

Taking the fixed plane as the plane of xy, let $x\,y\,z$ be the co-ordinates of a point in the body with regard to a fixed origin, and $x'\,y'\,z'$ the co-ordinates of the same with regard to the point in the body which is in contact with the plane. Let, also, ξ, η be

the co-ordinates of this latter point with regard to the fixed origin. We have then

$$x = x' + \xi, \ y = y' + \eta, \ z = z' + \zeta,$$

where ζ is evanescent.

$$\frac{d^2x}{dt^2} = \frac{d^2x'}{dt^2} + \frac{d^2\xi}{dt^2}, \ \frac{d^2y}{dt^2} = \frac{d^2y'}{dt^2} + \frac{d^2y}{dt^2}, \ \frac{d^2z}{dt^2} = \frac{d^2z'}{dt^2} + \frac{d^2\zeta}{dt^2}.$$

But, since the point $\xi\ \eta$ is, for the moment, at rest, and since, moreover, its first motion is perpendicular to the given plane, and changes sign at the instant of contact, we must have

$$\frac{d^2\xi}{dt^2} = 0, \ \frac{d^2\eta}{dt^2} = 0, \ \frac{d^2\zeta}{dt^2} = 0.$$

Whence

$$\frac{d^2x}{dt^2} = \frac{d^2x'}{dt^2}, \ \frac{d^2y}{dt^2} = \frac{d^2y'}{dt^2}, \ \frac{d^2z}{dt^2} = \frac{d^2z'}{dt^2}. \qquad (34)$$

These equations evidently contain the proposition, which may be otherwise enunciated thus:

The absolute accelerating force, at any point of a body which rolls without sliding, is equal and opposite to the relative accelerating force at the point of contact, estimated with regard to the first point as origin.

PROP. IV.

10. If a solid body roll without sliding upon a fixed rough plane, to determine the force of resistance of the plane at the point of contact.

Let it be supposed that in Prop. III. the first point is the centre of gravity of the rolling body. Then we know that the relative motion of the body with regard to this point is the same as if it were fixed—all the forces remaining unchanged;* and,

* It must be carefully observed that among these forces, whose values are supposed to remain unchanged, the resistance of the rough surface is included. It is not meant that the motion of a rolling body round its centre of gravity is actually the same, whether the centre of gravity be or be not fixed. This would evidently be untrue. If the geometrical condition of the system—namely, that the rolling

therefore, that the relative accelerating force at any point of the body, with regard to the centre of gravity, is not altered by conceiving this latter point to be fixed, and thereby converting *relative* into *absolute* accelerating force. Hence, and from the principle just stated, it is evident that the absolute accelerating force at the centre of gravity is equal and opposite to the absolute accelerating force which would exist at the point of contact, if the centre of gravity were considered, for the instant, as a fixed point.

Let $X\,Y\,Z$ be the components of the acting force at any point of the body, referred to the system of principal axes passing through the centre of gravity. Assume

$$\overline{X} = \int X dm, \quad \overline{Y} = \int Y dm, \quad \overline{Z} = \int Z dm.$$

Let, also, X', Y', Z' be the components of the resistance of the rough surface referred to the same axes. Then, the mass of

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body shall remain continually in contact with a fixed surface, be supposed to continue, the motion of this body round its centre of gravity would, in general, be changed by fixing this centre. So also would the resistance of the supporting surface, and, therefore, the condition given in the text would not be fulfilled. Fully stated, the assertion of the unchangeability of the motion round the centre of gravity is as follows:—

Conceive a second body identical with the rolling body in all respects but one—namely, that the supporting surface is replaced by a force equal in every respect to the resistance which the surface actually exerts upon the rolling body. Conceive, also, a third body identical with the second body in all respects but one—namely, that its centre of gravity is fixed. Then we know that the motion of this third body round its centre of gravity is identical with that of the second; and, therefore, that all the equations and formulæ connected with the motion of a solid body round a fixed point are applicable to the motion of the second body round its centre of gravity. But the equations of motion of the second body are, as is evident from the ordinary methods of mechanics, absolutely identical with those of the first. Hence the equations and formulæ alluded to are applicable to the motion of the first body.

All difficulty of conception will be removed, if we bear in mind that the third is an imaginary body subjected to *one* geometrical condition—namely, the fixity of the centre of gravity, and acted on by a force equal in every respect to the resistance which the supporting surface exerts upon a body *not* subject to this condition.

the body being unity, the components of the absolute accelerating force at the centre of gravity are

$$\overline{X} + X', \quad \overline{Y} + Y', \quad \overline{Z} + Z'.$$

Again, if x', y', z' be the co-ordinates of the point of contact with regard to the same axes, and p, q, r the components of the angular velocity, it is known that the components of the accelerating force which would exist at the point of contact, if the centre of gravity were fixed, are

$$\begin{aligned}
&z'\frac{dq}{dt} - y'\frac{dr}{dt} + (py' - qx')\,q + (pz' - rx')\,r \\
&x'\frac{dr}{dt} - z'\frac{dp}{dt} + (qz' - ry')\,r + (qx' - py')\,p \qquad (35) \\
&y'\frac{dp}{dt} - x'\frac{dq}{dt} + (rx' - pz')\,p + (ry' - qz')\,q.
\end{aligned}$$

These three components are, by the principle stated above severally equal to

$$-(\overline{X} + X'), \quad -(\overline{Y} + Y'), \quad -\overline{Z} + Z').$$

But if A, B, C be the principal moments of inertia, and if we put

$$\overline{L} = \int (yZ - zY)\,dm, \quad \overline{M} = \int (zX - xZ)\,dm, \quad \overline{N} = \int (xY - yX)\,dm,$$

we have

$$\begin{aligned}
Adp + (C - B)qrdt &= (\overline{L} + y'Z' - z'Y')\,dt \\
Bdq + (A - C)prdt &= (\overline{M} + z'X' - x'Z')\,dt \qquad (36) \\
Cdr + (B - A)pqdt &= (\overline{N} + x'Y' - y'X')\,dt.
\end{aligned}$$

Substituting then the values of the differential coefficients in the expressions (35), and equating them severally to $-(\overline{X} + X')$, $-(\overline{Y} + Y')$, $-(\overline{Z} + Z')$, we have three linear equations for the determination of the three unknown quantities, X', Y', Z'. The general result is, however, too complicated to be of much interest.

11. As an example, let us consider the case of a sphere which rolls on a rough plane, the acting force passing through its centre. We have then

$$A = B = C, \bar{L} = 0, \bar{M} = 0, \bar{N} = 0.$$

Since, moreover, all axes through the centre of gravity are principal axes, we may take at the given instant the axes x', y', z', parallel and perpendicular to the supporting plane. We have then $x' = 0$, $y' = 0$, $z' = a$. We have also, since the motion is at the instant a rotation round an axis parallel to the plane of $x'y'$, $r = 0$. The first two expressions (35) become, therefore,

$$a\frac{dq}{dt}, \quad -a\frac{dp}{dt};$$

and the first two of the equations (36) become

$$\frac{dp}{dt} = -\frac{aY'}{A}, \quad \frac{dq}{dt} = \frac{aX'}{A}.$$

Substituting these values, and putting $A = k^2$ (where k is the radius of gyration), we have

$$\bar{X} + X' = -\frac{a^2X'}{k^2}, \quad \bar{Y} + Y' = -\frac{a^2Y'}{k^2}.$$

Hence

$$(k^2 + a^2)X' = -k^2\bar{X}, \quad (k^2 + a^2)Y' = -k^2\bar{Y}.$$

These equations denote that the effective force of friction is parallel and contrary to the component of the acting force parallel to the plane. The intensities of these two forces are to each other in the ratio $k^2 : k^2 + a^2$; or, in the present case, $2 : 7$. Hence it is evident that the motion of the centre of the sphere is the same as if the plane were smooth, and the acting force were diminished in the proportion $7 : 5$.

Thus, for example, if gravity be the sole acting force, the centre of a sphere which moves on an inclined plane sufficiently rough to prevent sliding, will describe a parabola similar to that described by a single particle on a smooth inclined plane if

$$7 \sin (\text{incl. of smooth plane}) = 5 \sin (\text{incl. of rough plane}).$$

II.—*Initial Motion of a Solid Body resting upon one or more Fixed Rough Surfaces.*

Prop. V.

12. If an indefinitely small movement be communicated to a solid body resting in a given position upon one or more fixed rough surfaces, the resistances of the supporting surfaces will be, in general, determinate.

Let x_1, y_1, z_1, be the co-ordinates, referred to axes fixed in space, of a certain determinate point in the solid body. Let, also, θ, ϕ, ψ be the polar angles determining the position, with regard to the same axes, of a system of rectangular axes fixed in the body. We may conveniently take the centre of gravity for the determinate point, and the principal axes which intersect there for the system of moveable axes.

Let x, y, z be the co-ordinates of the point of contact with one of the supporting surfaces, referred to the fixed axes, and x', y', z' the co-ordinates of the same point referred to the moveable axes. We have, then, from the equation of the supporting surface,

$$f(x, y, z) = 0; \tag{38}$$

and from the equation of the surface of the body itself,

$$F(x', y', z') = 0. \tag{39}$$

Let, also, α, β, γ be the direction cosines of the common normal at the point of contact referred to the fixed axes, and α', β', γ' the direction cosines of the same line referred to the moveable axes. We have then,

$$\left.\begin{aligned} \alpha &= V\frac{df}{dx}, \ \beta = V\frac{df}{dy}, \ \gamma = V\frac{df}{dz}, \\ \alpha' &= V'\frac{dF}{dx'}, \beta' = V'\frac{dF}{dy'}, \gamma' = V'\frac{dF}{dz'}, \end{aligned}\right\} \tag{40}$$

where

$$V = \left(\frac{df^2}{dx^2} + \frac{df^2}{dy^2} + \frac{df^2}{dz^2}\right)^{-\frac{1}{2}}, \quad V' = \left(\frac{dF^2}{dx'^2} + \frac{dF^2}{dy'^2} + \frac{dF^2}{dz'^2}\right)^{-\frac{1}{2}}.$$

But we know from the ordinary equations of transformation that x', y', z' may be expressed as functions of the nine quantities, $x, y, z, x_1, y_1, z_1, \theta, \phi, \psi$. Similarly, α', β', γ' may be expressed as functions of the six quantities, $\alpha, \beta, \gamma, \theta, \phi, \psi$.

Substituting these values in the equations (40), we have six equations, five of which are independent, between the quantities, $x, y, z, x_1, y_1, z_1, \alpha, \beta, \gamma, \theta, \phi, \psi$. Eliminating between these and the equations (38), (39), the quantities $x\,y\,z$, $\alpha\,\beta\,\gamma$, we have a single equation of the form

$$F(x_1, y_1, z_1, \theta, \phi, \psi) = 0. \tag{41}$$

Differentiating this equation twice, and making the first differential coefficients vanish, according to the principle stated in p. 106, we have

$$\frac{dF}{dx_1}\frac{d^2x_1}{dt^2} + \frac{dF}{dy_1}\frac{d^2y_1}{dt^2} + \frac{dF}{dz_1}\frac{d^2z_1}{dt^2} + \frac{dF}{d\theta}\frac{d^2\theta}{dt^2} + \frac{dF}{d\phi}\frac{d^2\phi}{dt^2} + \frac{dF}{d\psi}\frac{d^2\psi}{dt^2} = 0. \tag{42}$$

A similar equation will be found for each supporting surface.

Let f be the effective force of friction at the point of contact, and a, b, c its direction cosines. Let also N be the normal reaction of the supporting surface. Then, if m be the mass of the body, we have, from the motion of the centre of gravity,

$$\left.\begin{aligned} m\frac{d^2x_1}{dt^2} &= mX + \alpha N + af + \alpha' N' + a'f' + \&c., \\ m\frac{d^2y_1}{dt^2} &= mY + \beta N + bf + \beta' N' + b'f' + \&c., \\ m\frac{d^2z_1}{dt^2} &= mZ + \gamma N + cf + \gamma' N' + c'f' + \&c. \end{aligned}\right\} \tag{43}$$

N', &c., f', &c., being the forces of reaction and friction at the other points of contact.

Again, differentiating the well-known equations,

$$p = \sin\phi \sin\theta \frac{d\psi}{dt} - \cos\phi \frac{d\theta}{dt},$$

$$q = \cos\phi \sin\theta \frac{d\psi}{dt} + \sin\phi \frac{d\theta}{dt},$$

$$r = \frac{d\phi}{dt} - \cos\theta \frac{d\psi}{dt};$$

and making the first differential coefficients vanish, as before we have

$$\left.\begin{aligned} \frac{dp}{dt} &= \sin\phi \sin\theta \frac{d^2\psi}{dt^2} - \cos\phi \frac{d^2\theta}{dt^2}, \\ \frac{dq}{dt} &= \cos\phi \sin\theta \frac{d^2\psi}{dt^2} + \sin\phi \frac{d^2\theta}{dt^2}, \\ \frac{dr}{dt} &= \frac{d^2\phi}{dt^2} - \cos\theta \frac{d^2\psi}{dt^2}. \end{aligned}\right\} \qquad (44)$$

But from the equations of rotation round the centre of gravity applied to the commencement of the motion, we have

$$A\frac{dp}{dt} = L, \quad B\frac{dq}{dt} = M, \quad C\frac{dr}{dt} = N, \qquad (45)$$

L, M, N being the components of the statical moment of the acting forces, including the forces of resistance, and A, B, C, the principal moments of inertia. Substituting these values in the equations (44), we have three equations to determine the differential coefficients—

$$\frac{d^2\theta}{dt^2}, \quad \frac{d^2\phi}{dt^2}, \quad \frac{d^2\psi}{dt^2}.$$

Substituting the values of the several differential coefficients of the second order in the equation (42), we have one equation between the quantities N, f, a, b, c, N', f', &c. A similar equation will be found for each point of contact. If now the motion given to the body be a motion of slipping at each point of contact, a, b, c, a', &c., are known, and $f = \mu N$, $f' = \mu' N'$, &c. The only unknown quantities, therefore, are N, N', &c., which are fully determined by the equations which have been found.

But if the motion of the body be, at some one of the points of contact, a motion of pure rolling, f may not have its maximum value, and its direction in the tangent plane will not be given. This case requires, therefore, to be treated separately.

PROP. VI.

13. A solid body rolls without slipping on a rough surface, the velocity being indefinitely small. To determine the direction and magnitude of the effective force of friction at the point of contact.

We have given in Prop. IV. the method of determining the resistance which a rough surface exerts upon a body which rolls without slipping upon the surface of another body. The results there obtained may be adapted to the case of initial motion simply by making $p = 0$, $q = 0$, $r = 0$.

The equations which are obtained by equating the expressions (35) to

$$-(X' + \overline{X}),\ -(Y' + \overline{Y}),\ -(Z' + \overline{Z}),$$

respectively, become then

$$\begin{aligned} z'dq - y'dr + (X' + \overline{X})\,dt &= 0, \\ x'dr - z'dp + (Y' + \overline{Y})\,dt &= 0, \\ y'dp - x'dq + (Z' + \overline{Z})\,dt &= 0. \end{aligned} \tag{46}$$

Multiplying these equations by x', y', z', respectively, and adding, we have

$$+ X'x' + Y'y' + Z'z' = -(\overline{X}x' + \overline{Y}y' + \overline{Z}z'), \tag{47}$$

showing that the component of the force of resistance resolved along the radius vector is equal and opposite to the acting force resolved in the same direction.

Substituting for dp, dq, dr, from equations (36), p. 137, we should find, as in the case of finite motion, three linear equations for the determination of X', Y', Z', but the results—although, of course, simpler than in the general case—are still too complicated to be of much interest.

Example 1.

14. A cylinder of any form is laid upon a rough inclined plane with its axis horizontal, and is acted on only by gravity. Required to determine whether the initial motion be pure rolling or a combination of slipping and rolling.

In accordance with the principle stated in p. 123, it will be sufficient to determine whether a motion of pure rolling be possible. For we learn from that principle that pure rolling, if a *possible* motion, is always the *actual* motion. We shall, therefore, proceed to inquire what is the coefficient of effective friction necessary to produce a motion of pure rolling. If this be not greater than the coefficient of maximum friction, the motion is one of pure rolling.

It is not necessary for this purpose to have recourse to the general method of p. 132, as the present question admits of being completely solved by means of the theorem given in p. 131.

For if the motion be one of pure rolling, it is evident that it is a pure rotation round an axis through T (Fig. 20) perpendicular to the plane of the paper. Hence G, P, T are points on the circle described as in the theorem referred to.

Let R be the normal reaction at T, and F the force of effective friction, and put, as before, $p = GQ$, $q = GP$, and k = radius of gyration round G. Then i being the inclination of the plane, we have

$$\text{Statical moment round } G = Rq + Fp,$$

$$\text{Statical moment round } P = gq \cos i + Fp,$$

$$\text{Statical moment round } T = g(q \cos i + p \sin i).$$

Hence, by the above-mentioned theorem,

$$\frac{Rq + Fp}{k^2} = \frac{gq \cos i + Fp}{k^2 + q^2} = g\,\frac{q \cos i + p \sin i}{k^2 + p^2 + q^2}.$$

Solving these equations for R and F, we find

$$\left.\begin{aligned}(k^2 + p^2 + q^2)R &= g\,\{(k^2 + p^2) \cos i - pq \sin i\}\\ (k^2 + p^2 + q^2)F &= g\,\{(k^2 + q^2) \sin i - pq \cos i\}.\end{aligned}\right\} \quad (48)$$

Hence, if the initial motion be as here supposed, the coefficient of effective friction will be

$$\frac{F}{R} = \frac{(k^2 + q^2)\sin i - pq\cos i}{(k^2 + p^2)\cos i - pq\sin i}. \tag{49}$$

If this be not greater than the coefficient of maximum friction, the initial motion will be one of pure rolling. If it be greater than the coefficient of maximum friction, the motion will be combined rolling and slipping.

The motion will be one of slipping without rolling if G lie behind TP, and the angle GTP be equal to the angle of friction; and if, moreover, the condition

$$F > \text{or} = \mu R$$

be fulfilled, where F and R are to be replaced by the values found above.

If q be positive—that is to say, if the centre of gravity be, as in the figure, *in front of* the perpendicular TP, the greatest inclination for which an initial movement of pure rolling is possible, will be found by equating to zero the value of R. This will give

$$\tan i = \frac{k^2 + p^2}{pq}. \tag{50}$$

If q be negative—that is to say, if the centre of gravity be *behind* the perpendicular TP, there is no limit to the inclination for which an initial movement of pure rolling is possible, if the plane be sufficiently rough. Thus, if the coefficient of friction be equal to, or greater than

$$\frac{k^2 + q^2}{pq},$$

a motion of pure rolling is possible for all inclinations.

15. It appears a paradoxical result that a motion of pure rolling should be possible if the plane be vertical. In such a case it might be supposed that no pressure could exist between the cylinder and the plane, and, therefore, no friction. If this be

so, it is plain that the cylinder should fall freely without any rotation.

To explain this seeming paradox we must remark, in the first place, that, if the body begin to revolve, there will be developed instantaneously, before any finite velocity has been attained, a finite pressure at the point of contact. This finite pressure will, of course, be accompanied by a finite force of friction, and there will no longer be any difficulty as to the motion supposed.

Now, if the mechanical arrangement, by which the cylinder was supported before the motion commenced, was such as to cause no pressure at the point of contact, and if none be developed by the particular mode in which the body is set free; or, in other words, if the *initial* value of the pressure be zero, the cylinder will fall freely without rotation.

But, let it be supposed that the initial value of the pressure, and, therefore, of the possible force of friction, is finite. Then, how short soever be the time during which this force is supposed to act, a rotatory motion will commence, which will, as we have seen, instantaneously develop a finite pressure at the point of contact, and, therefore, a finite force of friction; thus rendering the supposed motion possible. Even if the cessation of the initial pressure be instantaneous, so also is the commencement of the rotatory motion, and, therefore, the development of a new pressure. In this case it does not seem possible to decide what is the nature of the initial motion.

The ambiguity which meets us here is one of those which are due to the abstractions of Rational Mechanics, and can be removed only by an examination of the problem as it really exists in nature. Before considering it in this point of view, however, we shall illustrate the same principle by another example—

Example 2.

16. A rigid, weightless, rod carries at one extremity a heavy particle, the other extremity resting against a rough vertical wall. If the system be left to itself without initial velocity, to

determine the motion, the loaded extremity being initially higher than the other.

Let AB (Fig. 21) be the wall, EP the rod, and EH a horizontal line. Let it be supposed that HEP is not greater than the angle of friction. Then, *if the rod begin to revolve round* E, there will be a finite pressure against the wall in the direction PE; and, as HEP is less than the angle of friction, there will be no motion at E, and the rod will continue to revolve round this point. Then, inasmuch as the pressure against the wall must always lie in the line PE, and as the angle HEP is constantly diminishing, there can be no *slipping* at E, and the motion will be a pure rotation round E until the pressure vanishes.

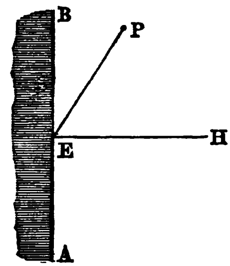

Fig. 21.

To find in what position of the rod this happens, put $HEP = \theta$, $EP = a$. Then, from the ordinary formulæ for the pressure on the point of support in the case of a simple pendulum, we find,

$$\text{Pressure (directed in } PE) = g \sin \theta - a\omega^2,$$

where ω is the angular velocity. But if θ_0 be the initial value of θ,

$$a\omega^2 = 2g (\sin \theta_0 - \sin \theta).$$

Hence,

$$\text{Pressure} = g (3 \sin \theta - 2 \sin \theta_0).$$

Equating this expression to zero, we have the value of θ, for which the motion ceases to be a rotation round E. Thus, for example, if the angle of friction $= 60°$, and if the rod be originally inclined to the horizon at this angle, the pressure will not vanish until the rod has revolved through an angle of nearly $25°$.

17. We see from this example that, *if the body begin to revolve*, it will revolve through a finite angle before the nature of the motion changes. But so long as we adhere to the abstractions of Rational Mechanics, we have no means of deciding whether or not it *will* begin to revolve. For, according to these ab-

stractions, the rod being regarded as absolutely incompressible, and, therefore, inelastic, the liberation of the body from the constraint by which it was prevented from moving, the cessation of the pressure corresponding to the state of rest, and the development of a new pressure corresponding to the state of motion, are absolutely simultaneous. It is, therefore, impossible to decide, so long as we preserve these abstractions, whether the initial motion be or be not a motion of rotation.

But this ambiguity is at once removed if we consider the rod as it really is—a body gifted with a certain amount of elasticity. If this be taken into account, it will be seen that the liberation of the body from restraint, and the cessation of the pressure corresponding to the state of rest, are *not* simultaneous. For this pressure, resulting from the compression of the rod, will not cease until the rod resume its natural state. If, then, the mechanical arrangements be so contrived that the rod shall be set completely free before it has resumed its natural state, the pressure will, at the commencement of the motion, still continue to exist, and the motion will be, as we have seen, a rotation round E. If, on the other hand, the mechanical arrangement be such that the rod shall have resumed its natural state before it is free to move, the pressure will, at the commencement of the motion, have ceased to exist, and the rod, with the attached weight, will fall freely without revolving.

Example 3.

18. Two equal cylinders are placed, with their axes horizontal, on a rough inclined plane, and are connected by a cord passing over the lower cylinder and under the higher cylinder, this cord being perpendicular to the axes. If the cylinders be now left to themselves, determine the initial motion.

Let the figure (Fig. 22) represent a section of the inclined plane and of the cylinders perpendicular to their axes, and let this section contain the cord. Let O be a fixed point, and assume $x = OA$, $x' = OA'$. Let also θ, θ' be the angles, estimated in the

direction of the arrows through which the cylinders have revolved since any fixed epoch. Then supposing the plane of the paper to bisect both the cylinders, and taking the common mass to be unity, we have from the motion of the centres of gravity the equations,

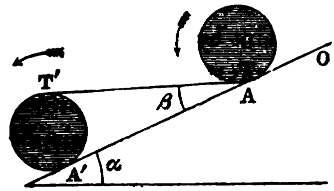

Fig. 21.

$$\frac{d^2x}{dt^2} = g \sin \alpha + T \cos \beta - F, \quad \frac{d^2x'}{dt^2} = g \sin \alpha - T \cos \beta - F', \quad (51)$$

where T is the tension of the cord, and F, F' are the effective forces of friction. Also, from the motion of the cylinders round their centres of gravity, we have

$$k^2 \frac{d^2\theta}{dt^2} = (F - T)\,a, \quad k^2 \frac{d^2\theta'}{dt^2} = (F' - T)\,a, \qquad (52)$$

k being the radius of gyration. Moreover, the amount of cord coiled round the cylinders at any time together with the length of the cord between them is constant. Now the amount of cord coiled round the two cylinders from the time of the commencement of the motion is

$$a\,(\theta + \theta' + 2\beta - 2\beta_0),$$

where β_0 is the initial value of β. Also, the length of the cord between the cylinders is

$$(x' - x) \cos \beta,$$

we have, therefore,

$$a\,(\theta + \theta' + 2\beta) + (x' - x) \cos \beta = \text{const.} \qquad (53)$$

whence, remembering that $(x' - x) \sin \beta = 2a$,

$$a\left(\frac{d\theta}{dt} + \frac{d\theta'}{dt}\right) + \cos \beta \left(\frac{dx'}{dt} - \frac{dx}{dt}\right) = 0. \qquad (54)$$

For the initial motion, differentiating a second time and rejecting all the first differential coefficients, we have

$$a\left(\frac{d^2\theta}{dt} + \frac{d^2\theta'}{dt^2}\right) + \cos\beta\left(\frac{d^2x'}{dt^2} - \frac{d^2x}{dt^2}\right) = 0. \qquad (55)$$

Substituting the values of these differential coefficients taken from the equations of motion, and putting $a^2 = n^2k^2$, we find

$$(n^2 - \cos\beta)\,F' + (n^2 + \cos\beta)\,F = 2\,(n^2 + \cos^2\beta)\,T. \quad (56)$$

We now proceed to consider the various kinds of motion of which the system is geometrically capable.

1. Both cylinders may roll without slipping. 2. The upper cylinder may roll without slipping, the lower cylinder rolling and slipping. 3. The lower cylinder may roll without slipping, the upper cylinder rolling and slipping. 4. Both cylinders may roll and slip. 5. Both cylinders may slip without rolling.

The last case is plainly impossible dynamically. For if there be no rotation we have, from the equations of motion, $F = T = F'$. But if P, P' be the pressures at the points of contact, we have, since both cylinders slip,

$$\left.\begin{aligned} F &= \mu P = \mu\,(g\cos\alpha - T\sin\beta) \\ F' &= \mu P' = \mu\,(g\cos\alpha + T\sin\beta). \end{aligned}\right\} \quad (57)$$

If, then, $F = F'$, we must have $T = 0$; and, therefore, $F = 0$, $F' = 0$, which is impossible if the plane be rough.

With respect to the other four cases, we know from the general principle (p. 123) that the first movement, if it be possible, will actually take place, and that the fourth will take place only when the other three are impossible. But the general principle does not give us any mode of deciding between the second and third. We shall see, however, that the second kind of movement is impossible.

(1). Both cylinders may roll without slipping.

In this case $dx = a\,d\theta$, $dx' = a'\,d\theta'$; we have, therefore, from equation (54),

$$(1 + \cos\beta)\frac{d\theta'}{dt} + (1 - \cos\beta)\frac{d\theta}{dt} = 0. \qquad (58)$$

It is plain from this equation that if both cylinders roll without slipping they must roll in opposite directions; also from the equations of motion, remembering that

$$\frac{d^2x}{dt^2} = a\frac{d^2\theta}{dt^2}, \quad \frac{d^2x'}{dt^2} = a\frac{d^2\theta'}{dt^2},$$

and putting $a^2 = n^2k^2$,

$$\left.\begin{aligned}(1 + n^2)F &= g\sin\alpha + (n^2 + \cos\beta)T\\ (1 + n^2)F' &= g\sin\alpha + (n^2 - \cos\beta)T.\end{aligned}\right\} \qquad (59)$$

Hence, as $n^2 > 1$, we must have

$$F > F', \quad F - T > F' - T.$$

But, inasmuch as the cylinders roll in opposite directions, one of the quantities, $F - T$, or $F' - T$, must be negative. The foregoing condition shows that $F' - T$ is the negative quantity. We see then that if both cylinders roll without slipping, the upper cylinder must roll *down*, and the lower cylinder *up*.

To determine the smallest coefficient of friction which will make this motion possible, it will be sufficient to determine this coefficient, so that there shall be no slipping at the point A. For we have already seen that $F > F'$; and it is plain that pressure at A < pressure at A'. Hence, coefficient of effective friction at A > coefficient of effective friction at A'. If, therefore, there be no slipping at A, there is, *a fortiori*, none at A'.

We have also, from equation (55), when both cylinders roll,

$$(1 + \cos\beta)\frac{d^2\theta'}{dt^2} + (1 - \cos\beta)\frac{d^2\theta}{dt^2} = 0;$$

whence, from equations (52),

$$(1 + \cos\beta)(F' - T) + (1 - \cos\beta)(F - T) = 0. \qquad (60)$$

Substituting from (59) for F, F', we have

$$(1 + \cos^2\beta)\,T = g\sin\alpha; \qquad (61)$$

whence

$$F = \frac{g \sin a}{1 + n^2}\left(1 + \frac{n^2 + \cos\beta}{1 + \cos^2\beta}\right). \qquad (62)$$

Also, if P be the pressure at A,

$$P = g\cos a - T\sin\beta.$$

Substituting in this expression the value of T, and putting $F = \mu P$, we obtain a value for μ which is the smallest coefficient of friction which will cause both cylinders to roll.

In order that this motion may be possible, it is necessary that the value for μ should be positive and finite. The foregoing value of F being always finite and positive, this condition will be fulfilled if $P > 0$. We have then

$$g\cos a - T\sin\beta > 0,$$

or

$$(1 + \cos^2\beta)\cos a > \sin\beta\sin a.$$

If β satisfy the equation

$$1 + \cos^2\beta_1 = \sin\beta_1\tan a,$$

all values of β lying between β_1 and 0 will satisfy the condition. Hence, whatever be the inclination of the plane, we can always find a coefficient of friction and a series of positions of the cylinders for which there will be perfect rolling. If $a =$ or $< 45°$, it is evident that for all finite values of β,

$$(1 + \cos^2\beta)\cos a > \sin\beta\sin a.$$

In this case, therefore, we can always find a coefficient of friction such that for *all* positions of the cylinder there will be perfect rolling.

(2). The upper cylinder may roll without slipping, the lower cylinder slipping and rolling.

It is plain that the slipping of the lower cylinder must be downwards. For, omitting friction, we have as the value of the accelerating force at A',

$$\frac{d}{dt}\left(\frac{dx'}{dt} - a\frac{d\theta'}{dt}\right) = g\sin a + (n^2 - \cos\beta)\,T.$$

This quantity being essentially positive, it is evident that the accelerating force at A' must be *downwards*. Hence, in con-

formity with the principle stated (p. 7), the force of friction must be *upwards*. The slipping, if it exist, must, therefore, be downwards. Hence the complete accelerating force at A' (including friction) must be positive, or

$$\frac{d}{dt}\left(\frac{dx'}{dt} - a\frac{d\theta'}{dt}\right) = g \sin a + (n^2 - \cos\beta)\, T - (n^2 + 1)\, F' > 0.$$

Hence

$$(n^2 + 1)\, F' < g \sin a + (n^2 - \cos\beta)\, T.$$

But if the upper cylinder roll, we have

$$(n^2 + 1)\, F = g \sin a + (n^2 + \cos\beta)\, T.$$

From these two conditions it appears that

$$F > F'.$$

But if P, P' be the pressures at A, A', we have, since the motion at A' is slipping, $F' = \mu P'$. Hence, as F cannot exceed μP, we must have $P > P'$. But this is impossible, inasmuch as $P = g \cos a - T \sin\beta$, $P' = g \cos a + T \sin\beta$, showing that necessarily $P < P'$. Hence it is impossible that the motion should be such as is here supposed.

(3). The lower cylinder may roll without slipping, the upper cylinder slipping and rolling.

Since the motion of the lower cylinder is pure rolling, we have from the second equation (59),

$$(1 + n^2)\, F' = g \sin a + (n^2 - \cos\beta)\, T;$$

and, since the upper cylinder slips, we must have

$$F = \mu P = \mu\,(g \cos a - T \sin\beta).$$

These two equations, together with the equation (56), are sufficient to determine F, F', T.

To determine the least value of μ which will make this movement possible, we have, in addition to the two preceding equations, and the general equation (56),

$$F' = \mu P' = \mu\,(g \cos a + T \sin\beta).$$

(4). If the value of μ be less than that derived from these equations, both the cylinders will slip and roll.

CHAPTER VI.

NECESSARY AND POSSIBLE EQUILIBRIUM.

1. We now proceed to consider the distinction, peculiar to our present subject, between positions in which equilibrium *may* exist, and positions in which it *must* exist. The former of these classes may be denominated positions of *possible* equilibrium, and the latter, positions of *necessary* equilibrium.

A position of *necessary* equilibrium, then, may be defined to be a position in which, if a system be placed in it without velocity, it will continue indefinitely.

A position of *possible* equilibrium is one in which a system may be so placed, without velocity, as to continue indefinitely, and yet may be so placed, still without velocity, as to move.

2. In order to understand the nature of this difference, it will be necessary to recur to a distinction already noticed—that, namely, between statical and dynamical friction.

It has long been known that the coefficient of maximum friction is not the same for the two different cases of rest and motion. Where one body rests upon another, the coefficient of maximum friction has a greater value than it would have if the two bodies were in a state of relative motion.

But there is a more important difference. When a material particle moves upon a rough surface, the direction of the force of friction is always directly opposed to that of the motion of the particle. If, therefore, this motion be governed by any geometrical condition which determines its line of direction on the supporting surface, the line of direction of the force of dynamical friction is also determined. Thus, for example, if the supporting surface be a plane, and if the particle be attached by a rigid rod to a fixed point, the position, and, therefore, the motion, of the particle, will necessarily be in a determinate circle. It is plain, therefore, that the line of direction of the frictional force

is necessarily determinate, being for every position of the particle, *while in motion*, a tangent to this circle.

3. More generally, if a system of particles whose positions are connected by certain geometrical relations, and each of which rests upon a rough surface, be in motion, we are not at liberty to assume for the direction of the force of friction at each point any line in the tangent plane at that point. For it is plain from what has been said, that a system of possible lines of *direction* for the forces of friction must be coincident with a system of possible lines of *motion* of the several particles. We shall now proceed to consider how far the directions of the forces of friction are restricted by this consideration.

Let $x_1 y_1 z_1$, $x_2 y_2 z_2$, ... $x_n y_n z_n$ be the co-ordinates of the several particles, and let it be supposed that these co-ordinates are connected by the equations,

$$L_1 = 0, \quad L_2 = 0, \text{ \&c.},$$

the number of these equations being m.

Let $\delta s_1, \delta s_2$, ... δs_n be a system of possible displacement of the particles; and let $a_1 b_1 c_1$, $a_2 b_2 c_2$, ... $a_n b_n c_n$ be the cosines of the angles which these displacements make with the axes. Then it is plain that we shall have the following equations of condition :—

$$\left(a_1 \frac{dL_1}{dx_1} + b_1 \frac{dL_1}{dy_1} + c_1 \frac{dL_1}{dz_1}\right)\delta s_1 + \left(a_2 \frac{dL_1}{dx_2} + \text{\&c.}\right)\delta s_2 + \text{\&c.} = 0$$

$$\left(a_1 \frac{dL_2}{dx_1} + \text{\&c.}\right)\delta s_1 + \text{\&c.} = 0, \qquad (1)$$

$$\text{\&c.}, \qquad \text{\&c.}$$

Now, if the number of these equations is sufficient to enable us to eliminate the $n - 1$ ratios,

$$\frac{\delta s_1}{\delta s_n}, \; \frac{\delta s_2}{\delta s_n}, \; \ldots \; \frac{\delta s_{n-1}}{\delta s_n},$$

we shall have $m - n + 1$ equations of condition between $a_1 b_1 c_1$, $a_2 b_2 c_2$, &c., and $x_1 y_1 z_1$, $x_2 y_2 z_2$, &c.

Hence we infer that,

If the number of equations connecting the positions of a system of material particles, each of which rests on a rough surface, be equal to, or greater than, the number of the particles, the lines of direction of the forces of dynamical friction cannot be assumed arbitrarily in the tangent planes at the several points, but are subject to certain equations of condition, the number of these equations being $m - n + 1$.

But even if $m < n$, in which case the *lines* of direction are arbitrary, the actual *directions* themselves are not so.

Thus, if AB, CD, EF, &c., be a system of possible directions for the forces of dynamical friction, we cannot take arbitrarily any combination of the signs in the series,

$$\pm AB \pm CD \pm EF, \text{ \&c.}$$

For, so long as there are any equations of the form (1), we cannot assign arbitrarily any combination of signs to the displacements δs_1, δs_2, &c. In fact, every equation of the form

$$A_1 \delta s_1 + A_2 \delta s_2 + \text{\&c.} = 0,$$

excludes two such combinations—those, namely, which make every term positive, and those which make every term negative. The same limitation, of course, holds for the signs of AB, CD, &c., which are opposed in direction to δs_1, δs_2, &c.

4. As an example of this principle, let us consider the case of a system of three material points, each resting on a rough surface, and connected with each other by three rigid inextensible rods. If the lengths of these rods be a, b, c, we have the equations of condition,

$$\begin{aligned}
(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 &= l_3^2 \\
(x_3 - x_1)^2 + (y_3 - y_1)^2 + (z_3 - z_1)^2 &= l_2^2 \qquad (2) \\
(x_2 - x_3)^2 + (y_2 - y_3)^2 + (z_2 - z_3)^2 &= l_1^2,
\end{aligned}$$

whence, by differentiation,

$$\begin{aligned}
(x_1 - x_2)(\delta x_1 - \delta x_2) + (y_1 - y_2)(\delta y_1 - \delta y_2) + (z_1 - z_2)(\delta z_1 - \delta z_2) &= 0 \\
(x_3 - x_1)(\delta x_3 - \delta x_1) + (y_3 - y_1)(\delta y_3 - \delta y_1) + (z_3 - z_1)(\delta z_3 - \delta z_1) &= 0 \quad (3) \\
(x_2 - x_3)(\delta x_2 - \delta x_3) + (y_2 - y_3)(\delta y_2 - \delta y_3) + (z_2 - z_3)(\delta z_2 - \delta z_3) &= 0.
\end{aligned}$$

Let, as before, δs_1, δs_2, δs_3 be the absolute displacements, and

let us represent by $\overline{l_3\delta s_1}$, $\overline{l_3\delta s_2}$, &c., the angles between the lines $l_3\, l_2\, l_1$, and the displacements δs_1, δs_2, δs_3.

Then it is easy to see that the foregoing equations may be written,

$$\begin{aligned}\delta s_1 \cos \overline{l_3\delta s_1} &= \delta s_2 \cos \overline{l_3\delta s_2}\\ \delta s_3 \cos \overline{l_2\delta s_3} &= \delta s_1 \cos \overline{l_2\delta s_1}\\ \delta s_2 \cos \overline{l_1\delta s_2} &= \delta s_3 \cos \overline{l_1\delta s_3},\end{aligned} \qquad (4)$$

δs_1, δs_2, δs_3, being the absolute displacements taken positively.

Hence, by eliminating δs_1, δs_2, δs_3, we have

$$\cos \overline{l_1\delta s_2} \cos \overline{l_2\delta s_3} \cos \overline{l_3\delta s_1} = \cos \overline{l_1\delta s_3} \cos \overline{l_2\delta s_1} \cos \overline{l_3\delta s_2}. \qquad (5)$$

This equation of condition must be satisfied by the directions of every system of possible movements δs_1, δs_2, δs_3; and since every system of possible directions, which can be assigned to the forces of dynamical friction, must coincide with a system, $-\delta s_1$, $-\delta s_2$, $-\delta s_3$, it is plain that the same equation of condition must be satisfied by the lines of direction of these forces.

Moreover, it is plain from the equations (4), that if the direction of δs_1 change sign, those of $\delta s_2\, \delta s_3$ must change sign also. Hence, if $+\delta s_2$, $+\delta s_2$, $+\delta s_3$ be a possible system, the only other possible combination of signs will be $-\delta s_1$, $-\delta s_2$, $-\delta s_3$.

5. As another example, if PQ (Fig. 23) be an inextensible rod laid upon the two rough rods, PA, QA, each of the extremities, P, Q, has two possible directions of movement, PA, AP, and QA, AQ; or, as it may be otherwise expressed, $\pm PA$, $\pm QA$. But of the four combinations included in $\pm PA$, $\pm QA$, two—namely, $+PA$, $+QA$, and $-PA$, $-QA$, are rendered impossible by the inextensibility of the rod. The remaining combinations, $+PA$, $-QA$, and $-PA$, $+QA$, are, therefore, the only possible directions of movement; and, consequently, $-PA$, $+QA$ and $+PA$, $-QA$ the only possible directions for the forces of dynamical friction.

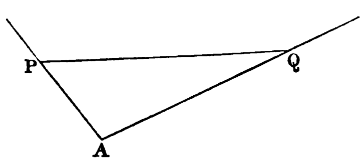

Fig. 23.

It appears from the foregoing discussion, that in every system connected by equations of condition, the existence of these

equations imposes certain restrictions upon the direction of the forces of dynamical friction. We shall now proceed to consider whether any such limitations exist in the case of statical friction.

6. The direction of the force of statical friction at each point of a system is governed by a rule analogous to, though not identical with, the rule which has been already stated for the case of dynamical friction. If, indeed, as we shall hereafter see, we discard the abstractions of Rational Mechanics, and state the conditions of each problem as they really exist in nature, these rules admit of a common statement. But, for the present, it is desirable to keep them distinct.

The force of statical friction—in other words, of that friction which is developed between two bodies which are in contact, the points of contact not having any relative motion—is, for each particle of a system which rests upon a rough surface, directly opposed to the tangential component of the resultant of all the other forces, external and geometrical, which act upon that particle. In order, therefore, to determine what limitations upon the directions of the forces of statical friction exist, we must inquire what limitation upon the direction of this resultant exists at each point.

(1.) If there be no equations of condition, and, therefore, no geometrical forces, the resultant of all the forces acting at each point (excluding friction) is perfectly definite both in direction and in magnitude. The direction of the force of statical friction is, therefore, also perfectly definite.

(2.) If there be m equations of condition (the number of particles being n), each of these equations will introduce a geometrical force, determinate in direction, but indeterminate in magnitude. The expressions for the directions of the resultant forces acting at each particle will, therefore, contain m indeterminate quantities. Since, then, the number of equations required for the determination of the directions of these forces is $2n$, the elimination of the m indeterminate quantities will leave $2n - m$ equations among these directions. The same number of equations will evidently exist between the directions of the tangential components of these resultant forces. Deducting from these the n equations which express the condition that each of these

components is situated in the tangent plane to the supporting surface, there will remain $n - m$ equations between their directions, and, therefore, between the directions of the forces of statical friction.

7. But these limitations upon the directions of the forces of statical friction do not in anywise interfere with the equilibrium of the system; in fact, they are themselves *conditions* of equilibrium. If the system can be kept in equilibrium at all, it will be so kept by forces whose directions satisfy these conditions. For it is evident that, when a system is in equilibrium, any one of the forces which act at each point must be directly opposed to the resultant of all the rest. The force of friction, therefore, which is one of these forces, and which necessarily acts in the tangent plane, must be directly opposed to the tangential component of the other forces. We have seen that this is precisely the direction which it does really assume. The force of friction, therefore, takes *of itself* the proper direction for equilibrium. So far, then, as the directions of these forces are concerned, the conditions of equilibrium are necessarily fulfilled. But for actual equilibrium it is further necessary, as we have before seen, that, at each particle, the resultant of all the forces, external and geometrical, shall lie within or upon the cone of resistance. If this condition be fulfilled, equilibrium will exist.

Now, it must be observed that the equations which express this condition are functions of the geometrical forces of the system, constituting a certain number of relations between these forces. Unless, therefore, the geometrical forces be such as to satisfy these equations, we cannot say that equilibrium actually exists. But the geometrical forces do not necessarily take such values as to satisfy these equations. In fact, as we shall see more clearly when we come to state the question without the abstractions of Rational Mechanics, the initial values of these forces are properly among the *data* of the problem. Their *directions* are, indeed, as we have before seen, determinate; but their *magnitudes* may be assigned arbitrarily. Unless, therefore, these magnitudes have been so assigned as to satisfy the conditions of equilibrium, the initial state of the system will be a state of motion.

8. We are now able to understand the meaning and reality of the kind of equilibrium spoken of at the commencement of this chapter—namely, *possible* equilibrium. The equilibrium of a system is *possible*, if we can assign to the geometrical forces such values as will keep the system at rest; *impossible*, if no such values, consistent with the conditions to which they are subject, can be assigned to these forces.

It remains that we inquire when the equilibrium of a system is *necessary*, and what are the conditions requisite to the actual existence of such a state of equilibrium. We shall, for the present, omit to notice the difference between the coefficients of friction for the state of rest and the state of motion, and shall conduct the investigation as if these coefficients were identical, reserving for future consideration the question, how far the results are modified by the difference between these coefficients.

9. The forces of statical friction take *of themselves*, as we have seen, the best directions for securing equilibrium—those, namely, which are, at each point, immediately opposed to the tangential component of the resultant of all the other forces. But these directions do not necessarily coincide with *any* system of directions which can be assigned to the forces of dynamical friction. These latter are not, therefore, necessarily the best directions for securing equilibrium, to which they may even be inclined at finite angles. It is, then, possible that a system may be in equilibrium under the influence of the forces of statical friction, and yet be removed by finite distances from the position in which it would be in equilibrium, if the friction became dynamical. In such a case, if the system be disturbed from its position of equilibrium by the communication of indefinitely small velocities to its several points, when the friction at each point will, of course, become dynamical, a *finite* force *tending to augment the displacement* may at once be developed at some or all of these points.*

* We have here a case differing in an important respect from ordinarily unstable equilibrium, where friction is not taken account of. In this latter case, if a number of indefinitely small velocities be communicated to the several points of the system, no finite force is called into play by the movement so given, and, therefore, the movement will not in any finite time become finite. Thus, for example, if a

Let it be supposed, then, that the initial values of the geometrical forces are inconsistent with the conditions of equilibrium. The system will then commence to move. Immediately on the commencement of this motion the force of friction at each moving point will become *dynamical;* that is to say, opposed in direction to the motion of the particle, and taking its maximum value. We have already shown (Prop. II., p. 105) that, in such a case, the geometrical forces acting upon the moving particles become determinate functions of the directions of the initial movements.

10. We are now enabled to determine the conditions of necessary (as distinguished from possible) equilibrium. For this purpose it will be necessary to change slightly the ordinary enunciation of the principle of virtual velocities. This principle may be enunciated as follows:—

A number of material particles, each of which rests upon a fixed surface, rough or smooth, are connected by given geometrical relations, and acted on by given forces, and are in equilibrium. Let x, y, z be the co-ordinates of one of these particles, and let X, Y, Z be the components of the resultant of all the forces, external, geometrical, and frictional, which act upon it. Conceive the system to be set in motion by the communication of an indefinitely small velocity to each particle. Let v be this velocity for the particle x, y, z, and a, b, c, the cosines of the angles which it makes with the axis. Then *if the forces X, Y, Z, do not change sensibly as the system passes from a state of rest to a state of motion*, the condition of equilibrium, necessary and sufficient, is

$$\Sigma mv\,(aX + bY + cZ) = \text{or} < 0,$$

in which we may understand X, Y, Z to refer either to the state of rest or to the state of motion, inasmuch as the forces are supposed to be the same for both states.

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material particle, placed on the highest point of a smooth surface, receive an indefinitely small velocity, this velocity, although it continually increases, will not in any finite time become finite. In the case of friction, on the contrary, where a *finite* force, tending to augment the displacement, may be called into play by an indefinitely small velocity, this velocity may in a finite time become finite.

If, on the other hand, any of the forces *do* change perceptibly in passing from a state of rest to a state of motion, we must understand by X, Y, Z in the foregoing condition the forces which act upon the particle *in the state of rest.*

Now, in such a case, it is quite possible that, although these forces may satisfy the foregoing condition, the forces which take their places, as the system passes from rest to motion, may not satisfy it. In such a case, then, if the system while at rest were acted on by the forces which do act upon it when in motion, the equilibrium could not exist.

Thus, for example, if the forces acting upon the particles when at rest be susceptible of a number of different systems of values, the forces which act upon the particles in motion being, as we have seen, absolutely determinate, it is possible that among the former systems of values might be found one or more to satisfy this condition, while, on the other hand, the single system corresponding to the state of motion might not satisfy it. In such a case equilibrium is *possible*, but not *necessary*.

If, however, the condition be true for all values of v, a, b, c, the forces X, Y, Z, having the values corresponding to the motion represented by these letters, equilibrium is *necessary*. For even if values be given to the forces inconsistent with this condition, these values, when the system begins to move, will at once be replaced by others which satisfy the condition. We shall then have a system acted on by forces which are in equilibrium, and whose several points are endued with indefinitely small velocities.

Now, if we adopt the abstractions of Rational Mechanics, the *vis viva* of such a system is plainly less than any assignable quantity. For, inasmuch as the forces corresponding to a state of rest are instantaneously changed into forces corresponding to a state of motion, which latter forces satisfy the condition,

$$\Sigma mv\,(aX + bY + cZ) < \text{or} = 0, \qquad (6)$$

and are, therefore, incapable of producing *vis viva*, the *time* during which the forces corresponding to the state of rest, which *are* capable of producing *vis viva*, act, is absolutely evanescent. Hence, whatever values be assigned to the forces of rest, if the

condition (6) be satisfied, the *vis viva* of the system can never become equal to any assignable magnitude. The system is, therefore, necessarily in equilibrium.

We infer, therefore—

1. *If it be possible to assign such values to the geometrical forces of the system, that the condition*

$$\Sigma m\,(X\delta x + Y\delta y + Z\delta z) = 0 \text{ or } < 0$$

be satisfied, the position is one of possible equilibrium.

2. *If this condition be satisfied by every system of values of the forces which corresponds to a system of possible movements, the equilibrium is necessary.*

11. As an example of the principle here laid down, let us consider the case of a material particle, R, (Fig. 24) connected by a rigid, weightless rod, with a fixed point, O, and resting against a rough vertical plane.

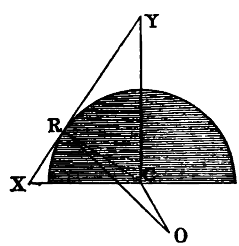

Fig. 24.

We have then, independently of friction, three forces acting upon the particle, namely—1. The weight g, whose direction is vertical. 2. The normal reaction, P, of the vertical plane. 3. The force, Q, arising from the attachment of the particle to the fixed point, acting in the direction OR. Then, if these forces be resolved parallel and perpendicular to the rough vertical plane, it is easily seen that the square of the former component will be, putting $ROC = \beta$, $RCY = \theta$,

$$Q^2 \sin^2\beta + g^2 - 2gQ \sin\beta \cos\theta.$$

This component must, for equilibrium, be destroyed by the force of friction. Now, if the particle be in an extreme position, the force of friction will be μP. But since the forces, resolved along the normal, must equilibrate

$$P = Q\cos\beta.$$

Since, then, the force of friction will, from its own nature, take a direction opposed to the above-mentioned component, the single condition of equilibrium will be

$$Q^2 \sin^2\beta + g^2 - 2gQ \sin\beta \cos\theta = \text{or} < \mu^2 Q^2 \cos^2\beta. \qquad (7)$$

This condition is impossible if

$$\sin^2\beta \cos^2\theta < \sin^2\beta - \mu^2 \cos^2\beta.$$

We have then, for the limiting position.

$$\sin\theta = \mu \cot\beta ;$$

an equation which gives the extreme obliquity of the radius RC, drawn to the particle.

If the obliquity be greater than this, equilibrium is impossible; but equilibrium does not necessarily exist, even when the obliquity lies within this limit, unless the geometrical force Q satisfy the condition (7). We must, therefore, in placing the system in its position of equilibrium, provide that the force, Q, shall satisfy this condition.

Let us now consider what are the conditions of *necessary* equilibrium.

In accordance with the principle stated (p. 160), let it be supposed that the particle receives a small velocity, whose direction will, of course, be a tangent to the vertical circle which is the locus of the point R. Of the two possible directions of displacement, RX, RY, it will evidently be sufficient to consider the former. For, if the equilibrium can be broken by an indefinitely small displacement, this will be effected rather by a displacement *downwards* than by a displacement *upwards*.

Then, taking CX, CY as axes, and denoting, as before, by Q the reaction of the rod, we have the equations of motion of the particle

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= Q(\sin\beta\sin\theta - \mu\cos\beta\cos\theta) \\ \frac{d^2y}{dt^2} &= Q(\sin\beta\cos\theta + \mu\cos\beta\sin\theta) - g. \end{aligned}\right\} \quad (8)$$

But by the geometry,

$$x = a\sin\theta,\ y = a\cos\theta.$$

Differentiating these equations twice, and rejecting, in accord-

ance with the general principle of p. 106, the square of the first differential coefficient, we have

$$\frac{d^2x}{dt^2} = a \cos\theta \frac{d^2\theta}{dt^2}, \quad \frac{d^2y}{dt^2} = - a \sin\theta \frac{d^2\theta}{dt^2}.$$

Substituting these values in the equations of motion, and dividing these equations one by the other, we find

$$Q \sin\beta = g \cos\theta.$$

Now, the accelerating force resolved in the direction of the displacement will be

$$\cos\theta \frac{d^2x}{dt^2} - \sin\theta \frac{d^2y}{dt^2} = g \sin\theta - \mu Q \cos\beta = g (\sin\theta - \mu \cot\beta \cos\theta).$$

If this force be positive, the velocity will increase, and, in a finite time, become finite. This will happen if

$$\tan\theta > \mu \cot\beta.$$

If the force be negative, the velocity will diminish, and, in an indefinitely short time, be destroyed. If the force be zero, the velocity will at first continue the same, but as its position changes, the force will increase, and the equilibrium will be broken. This, however, will not happen until after an indefinitely great length of time. These conditions are expressed by

$$\tan\theta > \mu \cot\beta, \ \tan\theta < \mu \cot\beta, \ \tan\theta = \mu \cot\beta.$$

Now, we have before seen that unless $\sin\theta > \mu \cot\beta$, the equilibrium is possible, if the value of Q be rightly assigned. It is plain, therefore, that between the positions θ_1 and θ_2 defined respectively by the equations

$$\tan\theta_1 = \mu \cot\beta, \text{ and } \sin\theta_2 = \mu \cot\beta,$$

equilibrium is *possible*, but not *necessary*.

The same discussion is made applicable to the case in which the original displacement is in the direction RY, by simply substituting $-\mu$ for μ, and resolving the accelerating force in the direction RY instead of the direction RX. The value of the force so resolved will be

$$-g (\mu \cot\beta \cos\theta + \sin\theta),$$

which, being essentially negative, will, in an indefinitely short time, reduce the particle to rest. But it is easily seen that between θ_1 and θ_2, the particle will not remain at rest. For it has been shown that there will not be equilibrium, unless the internal force, Q, satisfy the condition

$$Q^2 \sin^2\beta + g^2 - 2gQ \sin\beta \cos\theta < \text{or} = \mu^2 \cos^2\beta.$$

But it will be found that the value of Q is (as in the former part of the discussion) determined by the equation

$$Q \sin \beta = g \cos \theta.$$

Substituting this value in the preceding condition, we find

$$\tan\theta < \text{or} = \mu \cot\beta.$$

Hence it is evident that between the positions θ_1 and θ_2, if the particle receive a velocity in either direction the equilibrium will be broken.

With regard to the stability or instability of the equilibrium, it is necessary to make a distinction. Equilibrium may be called unstable when it is capable of being broken, either by the communication of an indefinitely small velocity, or by an indefinitely small change in the value of the coefficient of friction. These species of instability are not necessarily identical. Thus, in the present case, between the positions θ_1 and θ_2, the equilibrium is, as we have seen, necessarily unstable in the first sense. But whether or not it be unstable in the second sense, will depend upon the value which has been given to Q. For if the value of Q be such as to make $Q^2 + g^2 - 2gQ \sin\beta \cos\theta$ sensibly less than $\mu^2 Q^2 \cos^2\beta$, it is plain that μ may receive a sensible diminution without a rupture of the equilibrium. In this case the equilibrium is stable in the second sense. If, on the other hand, the value of Q be such as to make these expressions equal, the equilibrium is unstable in the second as well as in the first sense. The whole discussion may be summed up as follows:—

1. $\theta < \theta_1$. Equilibrium necessary and stable.

2. $\theta = \theta_1$. Equilibrium necessary and unstable in the first

sense, and may also be unstable in the second sense, if Q have the proper value.

3. $\theta > \theta_1 < \theta_2$. Equilibrium possible but not necessary. Unstable in the first sense, and may also be unstable in the second sense, if Q have the proper value.

4. $\theta = \theta_2$. Equilibrium possible and unstable in both senses.

5. $\theta > \theta_2$. Equilibrium impossible.

Prop. VII.

12. A solid body acted on by given forces rests upon two rough surfaces. Given the position of the body, to determine whether the equilibrium be necessary or only possible.

The principle of this investigation is identical with that which has been already applied to the case of a system of material particles. Here, as in the former case, we must inquire whether it be possible to give to the body such an indefinitely small movement as will give rise to a continually increasing *vis viva*. If this cannot be done, the equilibrium is necessary; if it can be done, the equilibrium is only possible.

Now, we have seen (pp. 139–42), that if a given indefinitely small motion be communicated to a solid body resting upon one or more rough surfaces, the resistances at the points of support are determinate. We have then a body acted on by given forces, and can calculate the increment of *vis viva* which these forces would produce. If this increment be negative, or indefinitely small, as compared with the small motion communicated to the body, the equilibrium is necessary. If neither of these suppositions be true, the equilibrium is only possible.

Let X, Y, Z, be the components of the resultant of all the forces applied to the body, and L, M, N, the components of the resultant moment. The parts of these forces and moments arising from the resistances of the supporting surfaces will, in general, as we have seen, depend upon the nature of the motion communicated to the body. Hence, supposing, as in p. 122, that the position of the body is determined by the variables x_1, y_1, z_1, ϕ, ψ, θ, and putting

$$dx_1 = avdt, \quad dy_1 = bvdt, \quad dz_1 = cvdt, \qquad (9)$$
$$d\phi = a\omega dt, \quad d\psi = \beta\omega dt, \quad d\theta = \gamma\omega dt,$$

the forces and moments are of the form

$$f(a, b, c, \quad \alpha, \beta, \gamma, v, \omega),$$

the form of the function being, in general, different for rolling and for slipping motion.

Now we know that the *vis viva* of a solid body = mass of body × (vel. of centre of gravity)^2 + *vis viva* corresponding to the rotation of the body round the centre of gravity. But the *vis viva* corresponding to this latter motion is (Poisson Mécanique, Tom. II., p. 150),

$$Ap^2 + Bq^2 + Cr^2.$$

Hence

$\frac{1}{2}$ inc. *vis viva* of solid body $= dV$

$$= \left\{ M \left(\frac{dx_1}{dt} \frac{d^2x_1}{dt^2} + \frac{dy_1}{dt} \frac{d^2y_1}{dt^2} + \frac{dz_1}{dt} \frac{d^2z_1}{dt^2} \right) + Ap\frac{dp}{dt} + Bq\frac{dq}{dt} + Cr\frac{dr}{dt} \right\} dt.$$

But, from the equations of motion of a solid body,

$$M\frac{d^2x_1}{dt^2} = X, \quad M\frac{d^2y_1}{dt^2} = Y, \quad M\frac{d^2z_1}{dt^2} = Z, \tag{10}$$

and

$$Ap\frac{dp}{dt} + Bq\frac{dq}{dt} + Cr\frac{dr}{dt} = Lp + Mq + Nr$$

$$= N\frac{d\phi}{dt} + \{(L\sin\phi + M\cos\phi)\sin\theta - N\cos\theta\}\frac{d\psi}{dt} \tag{11}$$

$$+ (M\sin\phi - L\cos\phi)\frac{d\theta}{dt},$$

substituting for p, q, r, from the general equations (3), p. 122.

Hence, and from equations (9) we find

$$dV = (aX + bY + cZ)\,vdt + \{N\alpha + (L\sin\theta\sin\phi + M\sin\theta\cos\phi$$

$$- N\cos\phi)\,\beta + (M\sin\phi - L\cos\phi)\,\gamma\}\,\omega dt. \tag{12}$$

If this value be zero or negative for all values of a, b, c, α, β, γ, v, ω, which are consistent with the conditions of the question, the equilibrium is necessary. Now, we have seen (p. 140), that, for each supporting surface, we have an equation of the form

$$F(x_1, y_1, z_1, \quad \phi, \psi, \theta) = 0.$$

Differentiating this equation, and substituting from (9), we have

$$v\left(a\frac{dF}{dx_1}+b\frac{dF}{dy_1}+c\frac{dF}{dz_1}\right)+\omega\left(\alpha\frac{dF}{d\phi}+\beta\frac{dF}{d\psi}+\gamma\frac{dF}{d\theta}\right)=0.$$

We have also

$$a^2+b^2+c^2=1,\quad \alpha^2+\beta^2+\gamma^2=1.$$

If by means of these equations as many as possible of the quantities a, b, c, &c., be eliminated from the value of dV, the condition of necessary equilibrium requires that, independently of the values of those which remain, we should have

$$dV = \text{or} < 0.$$

If the position of the solid body depend upon a single variable, u, two movements only are possible, represented by the two values $\pm\frac{du}{dt}$. In this case the condition of necessary equilibrium requires that the sign of the second differential coefficient $\frac{d^2u}{dt^2}$, as given by the equations of motion, should be (unless this coefficient vanish) different from the sign of $\frac{du}{dt}$.

Example 1.

13. Let $ABCD$ (Fig. 25) be a rectangular drawer, and let a force be applied to one of its handles, H, in the direction EH. Determine the condition and nature of equilibrium, if it exist.

It is evident that the drawer may be considered to be acted upon by three forces, namely, the moving force at H, and the two resistances of the rough sides at A and C. If equilibrium exist these three forces must meet in the same point. Draw the lines Aa, Aa', Cc, Cc', each making, with the tranverse sides AD, BC, an angle equal to the angle of friction. Then we know that the

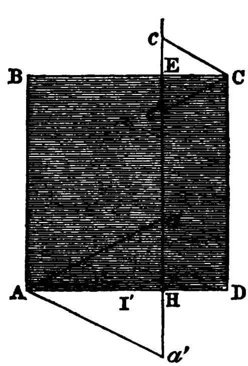

Fig. 25.

directions of resistance at the points A, C, must be respectively within the angles aAa', cCc'. If, then, as in fig. 25, the lines Aa, Aa', and the lines Cc, Cc', do not intercept any common portion of the line HE, it will be impossible to assign directions to the forces of resistance meeting at the same point of this line. In this case equilibrium is impossible. It is plain that this will happen, if the angle of friction be less than CAD. If, as in Fig. 26, the angle of friction be greater than CAD, it will always be possible to assign such directions and magnitudes to the forces of resistance at A and C, that their resultant shall be equal and opposite to the acting force. For this purpose it is only necessary to take any point p,

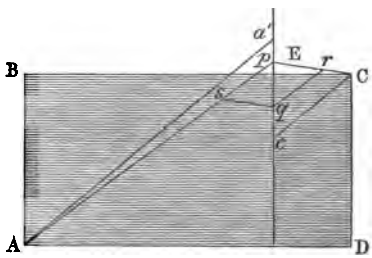

Fig. 26.

in the line ca', and join it with the points A, C. These joining lines Ap, Cp, are evidently possible directions of the forces of resistance. The magnitudes of these forces are found by taking a line pq to represent the acting force, and drawing the parallels qr, qs. The lines rp, sp, will represent the forces of resistance in magnitude and direction. It is plain then that the condition of possible equilibrium is

$$\tan CAD < \mu.$$

To determine the condition of necessary equilibrium let it be supposed that the drawer receives a small movement. The forces of friction will now assume their maximum values. Denoting then the applied force by F, the normal reactions at A and C by P, P', and taking the axes of co-ordinates parallel to AB, BC, the equations of motion of the centre of gravity of the drawer will be

$$m\frac{d^2x}{dt^2} = P - P' = 0$$

$$m\frac{d^2y}{dt^2} = F - \mu\,(P + P') = F - 2\mu P. \qquad (13)$$

But since there is no rotation, the sum of the moments of the acting forces with regard to the centre of gravity will be zero. Hence, putting (Fig. 25) $AB = 2a$, $IH = b$, we have

$$2Pa = Fb.$$

The accelerating force in the direction EH will, therefore, be

$$F\left(1 - \frac{\mu b}{a}\right).$$

If this force be positive, the small velocity communicated to the drawer will become finite, and the equilibrium will be permanently broken. If the force be negative, the drawer will return to rest in an indefinitely short time. If the force be zero, the drawer will simply retain the indefinitely small velocity which has been communicated to it. In neither of these two latter cases will the velocity ever become finite. The condition of necessary equilibrium is, therefore,

$$\mu b \text{ not} < a.$$

The whole result may, therefore, be stated as follows:—

If the coefficient of friction have any value from 0 to tan CAD, equilibrium is impossible.

If it have a value greater than tan CAD but less than $a \div b$, equilibrium is possible but not necessary.

If the coefficient of friction be greater than $a \div b$, equilibrium is necessary.

Example 2.

14. Two rods AB, CD, firmly connected at B, rest, as in fig. 27, upon a rough vertical board. Determine the limits of the inclination of AB to the vertical, and the nature of the equilibrium.

It is evident that the system of rods is kept in equilibrium by three forces, namely,—1. The weight acting in the line GV, G being the centre of gravity of the system. 2. The reaction of the rough board acting at the point E. 3. The reaction of the rough board acting at A. The only restriction by which these latter are bound is that their lines of direction shall not make with the perpendiculars to EB, EA, respectively, angles greater than the

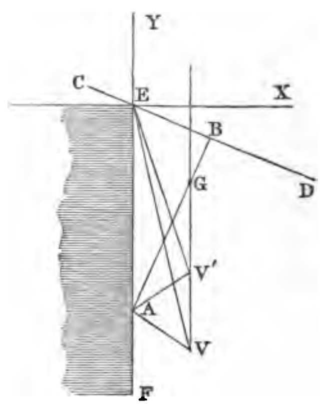

Fig. 27.

angle of friction. Let VE, AV, be these lines of direction respectively. Then it is necessary and sufficient for possible equilibrium, that these lines shall intersect upon the vertical GV. The extreme position consistent with possible equilibrium is, therefore, attained when two lines making the angles VAF, VEB, each equal to the complement of the angle of friction, intersect upon the line GV.

Let ϵ be the angle of friction, and assume $AB = a$, $AG = b$, $BAE = \theta$. Then evidently $AEV = \epsilon - \theta$, $AVE = 90° - (2\epsilon - \theta)$. Also $AE = a \sec \theta$, $AV = b \sin \theta \sec \epsilon$. But

$$AE \sin (\epsilon - \theta) = AV \cos (2\epsilon - \theta),$$

or by substitution

$$a \sec \theta \sin (\epsilon - \theta) = b \sin \theta \sec \epsilon \cos (2\epsilon - \theta), \quad (14)$$

an equation by which θ is determined.

Now it is plain, that of the two lines AV, AV', which can be drawn, each making with the horizontal line the angle of friction, the line AV, inclining *downwards*, corresponds to a more oblique position of the system than the line AV', inclining *upwards*.

For, since VE makes with the perpendicular to the rod at E a smaller angle than $V'E$, it is evident that if VE be in an extreme position, $V'E$ will be *beyond* the extreme position. Hence it appears that the extreme position of possible equilibrium is found by taking the direction of the force of friction at A *downwards*.

But now let it be supposed that the system receives a small velocity, so as to cause the point A to descend. The force of friction at A will at once commence to act *upwards*, and consequently the direction of the force of resistance at A will be AV'.

Taking now EX, EY, as axes of co-ordinates, we have as the equations of motion of the centre of gravity

$$\left.\begin{aligned} m\frac{d^2x}{dt^2} &= -R \sin (\epsilon - \theta) + R' \cos \epsilon, \\ m\frac{d^2y}{dt^2} &= R \cos (\epsilon - \theta) + R' \sin \epsilon - mg. \end{aligned}\right\} \quad (15)$$

R, R', being the complete resistances at E and A. Again, the equation of rotation round the centre of gravity will be

$$mk^2 \frac{d^2\theta}{dt^2} = R\{a \tan \theta \cos \epsilon - (a-b) \sin \epsilon\} - R'b \cos (\epsilon + \theta). \quad (16)$$

But by geometry,

$$x = b \sin \theta, \; y = -(a \sec \theta - b \cos \theta).$$

Differentiating these equations according to the principle stated, we have

$$\frac{d^2x}{dt^2} = b \cos \theta \frac{d^2\theta}{dt^2}, \qquad \frac{d^2y}{dt^2} = -(a \sec^2\theta + b) \sin \theta \frac{d^2\theta}{dt^2}.$$

But in order that the velocity may never become finite, it is necessary and sufficient that the value of $\frac{d^2\theta}{dt^2}$ should never become positive. The limiting position of necessary equilibrium is, therefore, found by making in (15), (16),

$$\frac{d^2\theta}{dt^2} = 0, \text{ and therefore } \frac{d^2x}{dt^2} = 0.$$

From these two equations we find, after reducing

$$b \sin \theta \cos^2\theta = a \cos \epsilon \sin (\epsilon - \theta), \qquad (17)$$

an equation which denotes that the lines AV', EV', meet on the vertical through the centre of gravity. As this is the condition of equilibrium when the resistance at A takes the direction AV' this result might have been foreseen.

It is easy to show that the equations (14) and (17) have each one, and but one, real positive root. Putting $\tan \theta = x$, $b = na$, we may write the first equation

$$x^3 + (2n - 1)\, x^2 \tan \epsilon + \{n(1 - \tan^2\epsilon) + 1\}x - \tan \epsilon = 0.$$

It is easily seen by the method of successive substitution that this equation has one real positive root between 0 and $\tan \epsilon$. But since b is always greater than $\frac{1}{2}a$, we have $2n - 1 > 0$. Hence, whatever be the sign of the third term, the equation will always have one variation and two permanences. There is, therefore, one and only one, real positive root.

If the equation (17) be similarly treated, it may be written

$$x^3 - x^2 \tan \epsilon + (n \sec^2\epsilon + 1)\, x - \tan \epsilon = 0.$$

It is evident that this equation has, like the former, one real positive root between o and tan ϵ. But if we denote the left hand side of this equation by u, we shall have

$$\frac{du}{dx} = 3x^2 - 2x \tan \epsilon + u \sec^2 \epsilon + 1.$$

It is easily seen that this quantity is always positive, hence u can only vanish once. There is, therefore, only one real root.

CHAPTER VII.

ON THE DETERMINATION OF THE ACTUAL VALUE OF THE ACTING FORCE OF FRICTION.

1. We have seen that in systems of bodies kept in equilibrium by friction, the actual amount of this force which is developed at each point of contact, is in certain cases indeterminate. Thus, in the case of a beam resting against two rough surfaces we may assume, consistently with the preservation of equilibrium, for the directions of the reaction of the supporting surfaces, any two lines situated on or within the cones of resistance, and intersecting on the vertical through the centre of gravity of the beam. It becomes, therefore, an interesting question to determine the actual amount of friction which is developed in any such case of equilibrium. In the phenomenon itself there can be nothing uncertain or indefinite, and if any such ambiguity appear in the solution which we have obtained, it must arise solely from some abstraction or simplification which we have introduced into the investigation, and which has no real existence in nature.

There is no difficulty in detecting the abstraction which rational mechanics introduces into questions such as that of the equilibrium of a beam resting upon two rough surfaces; it is the supposition of the perfect rigidity of the beam.

In the solution of all such problems, it is supposed that the beam preserves its original shape, and that there is no yielding on the part of the support which sustains it. If then the position be one of equilibrium, that is to say, a position in which no *finite* movement of the system ensues, it is supposed that the system, when placed in such a position, is at once reduced to a state of perfect rest. This, however, is not a true account of the phenomenon.

The force which in bodies resists a change of shape, being a function of that change vanishing with it, it is plain that, under

the influence of any distorting force, some change of shape will always ensue. Even then, when in one of those positions usually styled positions of equilibrium, the system will always begin to move, the resisting force being, at the commencement of the motion, absolutely zero.

2. Thus, when a heavy body is laid on a horizontal plane, the reaction of this plane is not an *instantaneous*, but a *progressive* force, commencing from zero. When the distance between the superficies of the body and the supporting plane has diminished to a certain value, both the body and the plane will begin to yield and change their shape.

From this change of shape there arises a force opposing the removal of the particles of each body from their original relative position, and increasing with the change of that position. Under the influence of these two opposing forces the body will continue to descend until they become equal, and, in virtue of its acquired velocity, for some time longer, and will finally be brought to rest when the velocity is distroyed by the preponderating force of resistance. If the body and the plane be perfectly inelastic, that is to say, if the force developed by distortion be merely a force tending to prevent further distortion, and not to restore the molecules to their original position, the phenomenon ends here; but if they be in any degree elastic, the developed force which has now increased above the force of gravity, will cause the body to rise, and then will ensue a series of small oscillations depending upon the elasticity of each substance.

3. Similarly, in the case of a beam resting on two rough surfaces, as soon as the beam comes in contact with the surfaces it commences to bend and contract under the influence of gravity, its own elasticity, and the resistance and friction of the surfaces, the last force always opposing itself to the motion of the extremities of the beam. Under the influence of these forces the beam will execute a series of small oscillations, the law of which will depend on that of the elastic force. If we know this latter force we can form the equations of the small oscillations of the beam, observing to change the sign of the force of friction whenever the motion of the point at which it is exerted changes its direction. If these equations can be integrated,

we shall know the position of the beam at the end of each oscillation, and therefore the value of the elastic force which depends on that position. Knowing this force in magnitude and direction at the two extremities of the beam, we know the pressure against the supports, and therefore (in magnitude and direction) the reaction of the surfaces. Changing now the sign of the force of friction at the extremity, which has for the moment ceased to move, we can determine whether the maximum value of the force of friction (with its sign thus changed), or any less value, would satisfy the condition of equilibrium at that point. If so, the point will cease to move. If not, a fresh oscillation will commence, and the investigation must be continued until we find the permanent position assumed by the beam. Knowing this position, and, therefore, the values of the elastic force or pressure at the two extremities, we know of course, both in magnitude and direction, the reactions of the fixed surfaces which are the objects of our inquiry.

The difficulty which attends the integration of the equations of motion of an elastic body, renders the complete solution of questions of this kind in general impossible, but the following case in which the solution is easily obtained will serve to exemplify the general method.

Example.

4. A heavy body is placed on a rough inclined plane, whose inclination is greater than $\tan^{-1}\mu$, and is supported by a string parallel to the plane, and attached to a fixed point. To determine the actual force of friction developed, and the tension of the string.

The only equation which statical science furnishes for the determination of these forces is

$$T + F = Mg \sin \alpha \qquad (1.)$$

leaving them both of course indeterminate, unless the value of the geometrical force T be given; or, in other words, unless the system be, not simply laid on the inclined plane, but strained by a given force, at the time when it is placed in its position of equilibrium. If this be so, and if the given force be within the

limits

$$Mg\,(\sin\alpha \pm \mu\cos\alpha)$$

equilibrium exists, and the actual value of F is given by eq. (1).

But if the system be simply laid on the plane, or, more generally, if the straining force lie without these limits, the problem ceases to be statical. The condition of equilibrium is not fulfilled, and the body begins to move. We shall suppose that there is no straining force, and that the original distance of the body from the fixed point is precisely the unstretched length of the string. We must now, in conformity with the principle stated in p. 175, replace the geometrical force, T, by the elasticity of the string, a force initially zero. The maximum force of friction being less than the resolved force of gravity, the body will begin to descend. Neglecting the elasticity of the body and of the plane, and assuming the elasticity of the cord to be proportional to its extension, divided by its original length, we have as the equation of motion of the body down the plane

$$M\frac{d^2x}{dt^2} = Mg\,(\sin\alpha - \mu\cos\alpha) - \frac{k^2x}{l}, \tag{2}$$

where l is the original length of the cord, and x is measured from the point at which the body would rest if the cord were drawn tight, but not extended.

Assuming for the sake of brevity

$$\theta^2p = g\,(\sin\alpha - \mu\cos\alpha),\ \theta^2 = \frac{k^2}{Ml},$$

we may write the equation

$$\frac{d^2(x-p)}{dt^2} + \theta^2(x-p) = 0.$$

Integrating this we have

$$x - p = A\cos\theta t + B\sin\theta t.$$

Suppose that at the beginning of the motion

$$x = 0,\ \frac{dx}{dt} = v',$$

we have

$$A = -p, \quad B = \frac{v'}{\theta}.$$

Hence

$$x = p\,(1 - \cos\theta t) + \frac{v'}{\theta}\sin\theta t. \tag{3}$$

The moment at which this motion ceases will be of course given by the equation

$$\frac{dx}{dt} = 0,$$

or

$$\tan\theta t = -\frac{v'}{\theta p}.$$

Substituting this value in (3) we have, recollecting that θt is in the second quadrant,

$$x = p + \sqrt{p^2 + \frac{v'^2}{\theta^2}}.$$

The corresponding value of the elastic force of the string will be

$$M\theta^2 x = M(\theta^2 p + \sqrt{\theta^4 p^2 + \theta^2 v'^2}).$$

Now, if this value be less than the *sum* of the resolved weight of the body, and the maximum force of friction estimated in the same direction, the body will remain at rest, and we have the equations

$$T = M(\theta^2 p + \sqrt{\theta^4 p^2 + \theta^2 v'^2}) \tag{4}$$

$$F = Mg\sin a - T,$$

to determine the actual values of the tension and the friction.

Suppose, for the sake of simplicity, that $v' = 0$, or that the body starts without initial velocity from the point at which the string is at its natural length. Then the foregoing equations give

$$T = 2M\theta^2 p = 2Mg(\sin a - \mu\cos a), \tag{5}$$

$$F = Mg\sin a - T = Mg(2\mu\cos a - \sin a).$$

Now, in order that the body may remain permanently at rest in this position, we must have

$$T < \text{or} = Mg\,(\sin a + \mu\cos a),$$

or

$$\tan \alpha < \text{or} = 3\mu.$$

If this condition be satisfied, the equations (5) will give the *permanent* values of the tension and the friction.

With respect to this latter force, it appears from the value which we have obtained—

1. That if $\tan \alpha < 2\mu$, the value of F will be *positive*, or the friction will act upwards.

2. That if $\tan \alpha = 2\mu$, the value of F is zero, or there is no friction developed.

3. That if $\tan \alpha > 2\mu$ and not $> 3\mu$, the value of F will be negative, or the friction will act downwards.

If $\tan \alpha > 3\mu$, the body will begin to ascend, and we shall have for its upward motion the equation

$$M \frac{d^2x}{dt^2} = Mg (\sin \alpha + \mu \cos \alpha) - \frac{k^2 x}{l},$$

or

$$\frac{d^2 (x - q)}{dt^2} + \theta^2 (x - q) = 0,$$

where

$$\theta^2 q = g (\sin \alpha + \mu \cos \alpha).$$

Integrating this as before, and determining the constants by the condition that at the commencement of the motion

$$x = 2p, \frac{dx}{dt} = 0,$$

we find

$$x = q + (2p - q) \cos \theta t. \tag{6}$$

From this equation, by the same reasoning as before, we find,

1. That if $\tan \alpha$ be not greater than 5μ, the body will come to a state of permanent rest at the end of this oscillation, when $\theta t = \pi$.

2. That if this condition be fulfilled, the permanent values of the tension and friction will be given by the equations,

$$T = 4\mu Mg \cos \alpha, \tag{7}$$
$$F = Mg (\sin \alpha - 4\mu \cos \alpha).$$

If $\tan \alpha$ be greater than 5μ, the body will descend, and the

same method must oe pursued until the body come permanently to rest. The result of the whole discussion of the case of a heavy body placed on a rough inclined plane, at a distance from the fixed point equal to the natural length of the supporting string may be summed up as follows :—

1. If tan a be greater than $(4n+1)\mu$, and not greater than $(4n+3)\mu$, the body will come permanently to rest at the end of the $(2n+1)^{th}$ semi-oscillation, and the values of the tension and friction will be

$$T = 2Mg\,\{\sin a - (2n+1)\mu \cos a\}. \qquad (8)$$

$$F = Mg\,\{(4n+2)\mu \cos a - \sin a\}.$$

2. If tan a be greater than $(4n+3)\,\mu$, and not greater than $(4n+5)\mu$, the body will come to rest at the end of the $(2n+2)^{th}$ semi-oscillation and the values of T and F will be

$$T = 4(n+1)\mu\, Mg \cos a,$$

$$F = Mg\{\sin a - 4(n+1)\mu \cos a\}. \qquad (9)$$

3. If tan a be any even multiple of μ, $F = 0$, and the tension is the same as if the plane were smooth.

4. If the value of tan a lie between $4n\mu$ and $(4n+2)\mu$, the value of F is positive, and the tension is *less* than if the plane were smooth.

5. If the value of tan a lie between $(4n+2)\mu$ and $(4n+4)\mu$, the value of F is negative, and the tension is *greater* than if the plane were smooth.

CHAPTER VIII.

MISCELLANEOUS PROBLEMS.

I.—*Problem of the Top.*

1. THE problem of the top may be generally stated as follows:—

A solid of revolution, to which has been communicated a very rapid motion round its axis of figure, is placed upon a rough horizontal plane. Required to determine its motion.

Let Fig. 28 represent the top, the plane of the paper being the vertical plane passing through the axis, and, therefore, through the point of contact. Let G be the centre of gravity of the top, and GZ_1 the axis of figure. Let also $QNQ'N'$, QZ_1Q' be the equatorial and meridional sections, respectively; the line QQ' being in the plane of the paper. Then the equatorial plane, $QNQ'N'$ is evidently perpendicular to the plane of the paper. The same is true of the horizontal plane

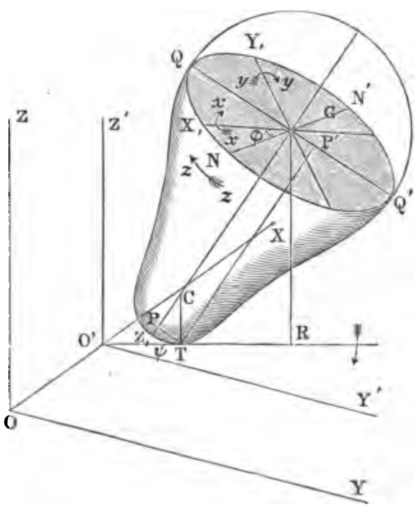

Fig. 28.

on which the top rests. Hence the line NN', which is parallel to the intersection of these planes, is perpendicular to the plane of the paper.

Let the origin of the fixed axes be taken in the horizontal plane, the axis of z being vertical; and let x, y, z be the coordinates of G. Let OX, OY, OZ be parallel to these axes.

Then if T be the point of contact of the top with the horizontal plane, it is evident that TR is perpendicular to the nodal line NN'. Let GX_1, GY_1, GZ_1 be the movable axes, and GR the perpendicular on the horizontal plane; also, let TP, TP' be perpendicular to GZ, QQ', respectively.

Then it is plain that the polar angles which determine the position of the body are

$$TO'Y' = \psi, \; NGX_1 = \phi, \; RGZ_1 = \theta.$$

We shall suppose that the lower extremity of the top is a spherical surface, of which C is the centre. If, then, we assume

$$\text{Radius of sphere} = TC = a, \; GC = b$$

$$TP = \xi, \; TP' = \eta, \; TR = z',$$

we have from the figure the following geometrical equations :—

$$\xi = a \sin\theta, \; \eta = b + a\cos\theta, \; z = a + b\cos\theta, \; z' = b\sin\theta. \quad (1)$$

Let R be the complete reaction of the horizontal plane, and ϵ the angle of friction.

We shall assume, for the purpose of simplifying the question, that the motion of rotation round the axis of figure is so rapid, as compared with either of the other angular motions of the body, that in determining the direction of the force of friction the motion of T, considered as a point in the body, may be taken to be perpendicular to the plane of the paper.

If, therefore, the rotation be in the direction of the arrow at N, the force of friction is parallel to the line NN', the velocity of rotation being so great that there is necessarily slipping at T. The complete reaction, R, may, therefore, be resolved into $R\cos\epsilon$ parallel to OZ, and $R\sin\epsilon$ parallel to NN'. The second of these components, which is the force of friction, may be further resolved into $R\sin\epsilon\cos\psi$, $R\sin\epsilon\sin\psi$, parallel to the axis of x and y, respectively.

To determine the components, with respect to the movable axes GX_1, GY_1, GZ_1 of the moment of R round the centre of gravity, we shall suppose this force to be replaced by its com-

ponents $R \cos \epsilon$, $R \sin \epsilon$. Now, the moment of $R \cos \epsilon$ with regard to G is $R \cos \epsilon \, XTR$, and the axis of the moment is NN'. This moment may be resolved into

$$R \cos \epsilon \times TR \cos \phi, \; R \cos \epsilon \times TR \sin \phi, \; 0$$

round GX_1, GY_1, GZ_1, respectively. Similarly, the moment of $R \sin \epsilon$ may be resolved into

$$R \sin \epsilon \times TP' \sin \phi, \; R \sin \epsilon \times TP' \cos \phi, \; R \sin \epsilon \times TP$$

round the same axis.

To fix the signs of these moments, it is to be remarked that if the angles ϕ, θ be measured as in the figure, the *positive* directions of the movable axes will be GX_1, GY_1, GZ_1. Now, since the components p, q, r of the angular velocity are proportional to the indices of position of the instantaneous axis with regard to these axes, it is evident that the *positive* directions of the rotations indicated by p, q, r must correspond to a position of the instantaneous axis lying within the solid angle made by the *positive* directions GX_1, GY_1, GZ_1. Hence it is easy to see that if the direction of the impressed rotation round GZ_1 be that indicated by the arrow zz, the positive directions of p and q will be indicated by the arrows xx, yy, respectively. The moment of the normal resistance $R \cos \epsilon$ will, therefore, be negative with respect to GX_1, and positive with regard to GY_1. The moment of the force of friction $R \sin \epsilon$ will be negative with regard to all three axes. If, then, LMN be the moments of the total forces with respect to the movable axes, we have

$$\begin{aligned} L &= -R(z' \cos \epsilon \cos \phi + \eta \sin \epsilon \sin \phi) \\ M &= -R(z' \cos \epsilon \sin \phi - \eta \sin \epsilon \cos \phi) \qquad (2) \\ N &= -R\xi \sin \epsilon. \end{aligned}$$

2. We now proceed to form the differential equations of motion. We have, in the first place, for the motion of the centre of gravity,

$$\begin{aligned}
\frac{d^2x}{dt^2} &= R \sin \varepsilon \cos \psi \\
\frac{d^2y}{dt^2} &= R \sin \varepsilon \sin \psi \qquad (3) \\
\frac{d^2z}{dt^2} &= R \cos \varepsilon - g
\end{aligned}$$

the mass of the body being taken = 1.

Also, for the motion of rotation round the centre of gravity, substituting the values of L, M, N derived from (2), in the general equations (2), p. 122, we find

$$\begin{aligned}
Ap' - (A - C)\,qr &= -R\,(z' \cos \varepsilon \cos \phi + \eta \sin \varepsilon \sin \phi) \\
Aq' + (A - C)\,pr &= R\,(z' \cos \varepsilon \sin \phi - \eta \sin \varepsilon \cos \phi) \qquad (4) \\
Cr' \qquad\qquad &= -R\xi \sin \varepsilon,
\end{aligned}$$

where we have put, for the sake of brevity,

$$dp = p'dt,\ dq = q'dt,\ dz = z'dt.$$

We have also the general equations,

$$\begin{aligned}
p &= \psi' \sin \theta \sin \phi - \theta' \cos \phi \\
q &= \psi' \sin \theta \cos \psi + \theta' \sin \phi \qquad (5) \\
r &= \phi' - \psi' \cos \theta;
\end{aligned}$$

putting in like manner,

$$d\psi = \psi'dt,\ d\phi = \phi'dt,\ d\theta = \theta'dt.$$

3. Assuming

$$A - C = nA,\ udt = \sin \theta d\psi,\ u'dt = du,$$

we find from the first two equations (4),

$$\begin{aligned}
A\,\{ q' \sin \phi - p' \cos \phi + nr\,(p \sin \phi + q \cos \phi) \} &= Rz' \cos \varepsilon \\
&\qquad\qquad (6) \\
A\,\{ q' \cos \phi + p' \sin \phi + nr\,(p \sin \phi - q \cos \phi) \} &= -R\eta \sin \varepsilon.
\end{aligned}$$

Also, from the first two equations (5) we have

$$p \sin \phi + q \cos \phi = u,\ p \cos \phi - q \sin \phi = -\theta'.$$

Whence, by differentiation,

$$p' \sin \phi + q' \cos \phi - \theta'\phi' = u'$$

$$p' \cos \phi - q' \sin \phi - u\phi' = - \theta'';$$

or, replacing ϕ' by its value given by the third equation (5) namely, $\phi' = r + u \cot \theta$,

$$p' \sin \phi + q' \cos \phi = u' + \theta' (r + u \cot \theta)$$

$$p' \cos \phi - q' \sin \phi = u (r + u \cot \theta) - \theta''.$$

Introducing these values into the equations (6), and replacing n by its value, we find

$$\left. \begin{array}{l} A\theta'' - Cru - Au^2 \cot \theta - Rz' \cos \epsilon = 0 \\ A (u' + u\theta' \cot \theta) + Cr\theta' + R\eta \sin \epsilon = 0. \end{array} \right\} \quad (7)$$

Multiplying the second of these equations by $\sin \theta dt$, and integrating, we have

$$Au \sin \theta - Cr \cos \theta + C \int \cos \theta dr + \sin \epsilon \int R\eta \sin \theta dt = \text{const.}$$

or, substituting for Rdt from the third equation (4),

$$Au \sin \theta - Cr \cos \theta + C \int (\cos \theta - \frac{\eta}{\xi} \sin \theta) \, dr = \text{const.} \quad (8)$$

But from the geometrical relations (1) we find

$$\cos \theta - \frac{\eta}{\xi} \sin \theta = - \frac{b}{a} = - i \text{ (suppose)}.$$

Substituting this value in (8), we have

$$Au \sin \theta - Cr \cos \theta - iCr = \text{const.}$$

To determine the constant, let it be supposed that the only velocity originally communicated to the top is that of rotation round the axis of figure. Let r_0 denote this velocity, and let θ_0 be the original value of θ. We have then, replacing u by its value, the following first integral of the differential equations of the motion of the top :—

$$A \sin^2 \theta \frac{d\psi}{dt} + C (r_0 \cos \theta_0 - r \cos \theta) + iC (r_0 - r) = 0. \quad (9)$$

4. It may readily be shown from this integral that the axis of the top will soon become vertical, assuming, as before, that

the other motions are slow as compared with the rotation round the axis. For it is plain from the third equation (4), that the action of the friction is constantly to diminish r. We have then necessarily, $r < r_0$. Let this diminution have progressed so far that

$$(i + 1)\, r = (i + \cos\theta_0)\, r_0. \tag{10}$$

Substituting this value of r in (9), we find

$$A \sin^2\theta \frac{d\psi}{dt} + \frac{Cr_0\,(i + \cos\theta_0)\,(1 - \cos\theta)}{i + 1} = 0,$$

which may be put under the form

$$\sin^2 \frac{1}{2}\theta \left\{ 2A \cos^2 \frac{1}{2}\theta \frac{d\psi}{dt} + Cr_0\left(\frac{i + \cos\theta_0}{i + 1}\right) \right\} = 0. \tag{11}$$

Now, it is easily seen that, under the suppositions which we have made above, the second of these factors cannot vanish. For, in the case of a top, i is a large number. Hence the coefficient of Cr_0 is $q.p.$ unity. Unless, therefore, either the velocity of the node be comparable with that of rotation, or the polar moment of inertia be very small as compared with the equatorial moment, which is not true in the case of the top, the second factor is necessarily positive. We must, therefore, have $\theta = 0$, showing that the axis becomes vertical when the velocity of rotation satisfies the equation (10).

The diminution of velocity indicated by this equation is in general small, even when the original position of the axis is very oblique. For we have

$$r_0 - r = r_0\left(\frac{1 - \cos\theta_0}{i + 1}\right).$$

Suppose the distance of the centre of gravity from the lower extremity of the top to be two inches, and the radius of the spherical end to be one-tenth of an inch, and let $\theta_0 = 60°$; we have then,

$$i + 1 = 20, \quad 1 - \cos\theta_0 = \cdot 5.$$

Hence,

$$r_0 - r = \frac{r_0}{40};$$

showing that the axis will rise to a vertical position as soon as one-fortieth of the original velocity is destroyed.

In general it is plain that this elevation of the axis to a vertical position will be accomplished with a loss of velocity which will vary directly as the radius of the spherical end, and inversely as the distance of the centre of gravity from this end.

It is evident, from the first equation (7), that when $t = 0$, and therefore (by the initial conditions), $u = 0$, $\theta'' > 0$. Hence, inasmuch as, initially, $\theta' = 0$, θ' will at first become positive; or, in other words, the axis will commence by descending. But it appears from the second equation (7), that this downward velocity cannot attain any sensible value. For if any sensible *positive* value were assigned to θ', it is plain that, on account of the magnitude of r, the left side of the equation would necessarily become positive, and could not, therefore, vanish.

5. If, in the third equation (3), we substitute for z its value $a + b\cos\theta$, and eliminate R by means of the first equation (7), we find

$$(A + b^2 \sin^2\theta)\frac{d^2\theta}{dt^2} + b^2 \sin\theta\cos\theta\frac{d\theta^2}{dt^2} - bg\sin\theta = u(Cr + Au\cot\theta)$$

This equation may be written,

$$\frac{d}{d\theta}\left\{(A + b^2\sin^2\theta)\frac{d\theta^2}{dt^2} + 2bg\cos\theta\right\} = 2(Cr + Au\cot\theta)u. \quad (12)$$

Eliminating r from this equation by means of (9), we have an equation only involving θ, u, and t. A second equation between the same variables is obtained by eliminating R between the equations (7).

But these equations do not seem to admit of further integration.

II.—*Friction Wheels.*

6. A wheel is attached firmly to an axle so as to turn with it. The system is then placed, as in Fig. 29, so that the axle rests

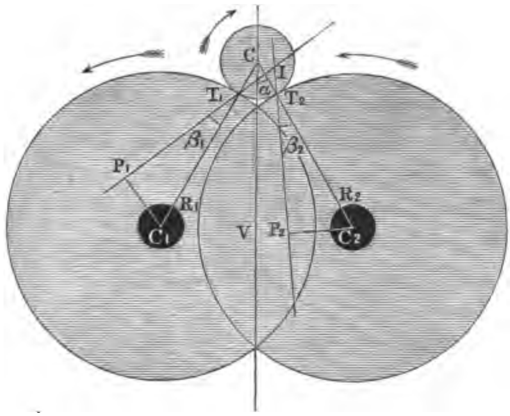

Fig. 29.

upon four vertical wheels, whose axes rest upon fixed bearings. If the first wheel have a given velocity of rotation, to compare the length of time which will elapse before it come to rest, with the duration of the movement which would have taken place if the wheel itself had rested on the fixed bearings.

Let the fixed bearings be four cylindrical holes, and let the supporting wheels be equal in every respect. Let Fig. 29 represent a section of the entire system by a plane perpendicular to the axles, and cutting two of the supporting wheels—the directions of the rotation being indicated by the arrows. Then it is plain that the directions of the forces of friction acting upon the axle, C, at T_1, T_2, will be in both cases from left to right. Hence, if T_1I, T_2I be the directions of the complete reactions of the supporting wheels on the axle C, these lines must be, as in the figure, to the *right* of the corresponding radii CT_1, CT_2.

Put

$$C_1T_1 = C_2T_2 = a, \; CT_1 = CT_2 = b, \; C_1R_1 = C_2R_2 = r;$$

and let $2W$ be the weight of the supported wheel and axle. Put also $$VCC_1 = VCC_2 = \alpha.$$

Let R_1, R_2 be the normal reactions at T_1, T_2, and F_1, F_2 the

forces of friction at these points. Then, since C remains at rest, and since, moreover, $2W$ is equally supported by each pair of wheels, we have

$$\left.\begin{aligned}(R_1 + R_2)\cos a - (F_1 - F_2)\sin a &= W\\ (R_1 - R_2)\sin a + (F_1 + F_2)\cos a &= 0.\end{aligned}\right\} \quad (13)$$

We assume that each supporting wheel bears upon each of its cylindrical supports in one line only. This assumption is necessarily true, if the cylindrical hole be in the smallest degree larger than the axle which turns in it. Let η_1, η_2 be the angles which planes passing through these lines and the axes of the wheels, respectively, make with the vertical. Let also P_1, P_2 be the normal reactions of two of the cylindrical bearings. Then, since the axes of the wheels remain at rest, and since the forces of friction exerted by these bearings have evidently their maximum values, we have

$$\left.\begin{aligned}R_1\cos a - F_1\sin a + W' &= 2P_1(\cos\eta_1 + \mu\sin\eta_1) = 2P_1\cos(\eta_1 - \varepsilon)\sec\varepsilon\\ R_1\sin a + F_1\cos a &= 2P_1(\sin\eta_1 - \mu\cos\eta_1) = 2P_1\sin(\eta_1 - \varepsilon)\sec\varepsilon\\ R_2\cos a + F_2\sin a + W' &= 2P_2(\cos\eta_2 - \mu\sin\eta_2) = 2P_2\cos(\eta_2 + \varepsilon)\sec\varepsilon\\ R_2\sin a - F_2\cos a &= 2P_2(\sin\eta_2 + \mu\cos\eta_2) = 2P_2\sin(\sin\eta_2 + \varepsilon)\sec\varepsilon,\end{aligned}\right\} \quad (14)$$

where W' is the weight of each of the supporting wheels.

Assume, for the sake of brevity,

$$\eta_1 - \varepsilon = \theta, \quad \eta_2 + \varepsilon = \phi.$$

N. B.—It is plain that θ, ϕ, are the angles which the complete reactions of the fixed bearings make with the vertical.

Solving these equations (14) for R_1, R_2, F_1, F_2, we find

$$\left.\begin{aligned}R_1 &= 2P_1\sec\varepsilon\cos(\theta - a) - W'\cos a\\ F_1 &= 2P_1\sec\varepsilon\sin(\theta - a) + W'\sin a\\ R_2 &= 2P_2\sec\varepsilon\cos(a - \phi) - W'\cos a\\ F_2 &= 2P_2\sec\varepsilon\sin(a - \phi) - W'\sin a.\end{aligned}\right\} \quad (15)$$

Substituting these values in (13), we have

$$2(P_1\cos\theta + P_2\cos\phi) = (W + 2W')\cos\varepsilon, \quad P_1\sin\theta = P_2\sin\phi;$$

whence

$$P_1 = \frac{(W + 2W') \sin\phi \cos\varepsilon}{2\sin(\phi + \theta)}, \quad P_2 = \frac{(W + 2W') \sin\theta \cos\varepsilon}{2\sin(\phi + \theta)}. \quad (16)$$

Substituting these values in (15), we have, putting $W + 2W' = nW$,

$$R_1 = \frac{nW \sin\phi \cos(\theta - a)}{\sin(\phi + \theta)} - W' \cos a$$

$$F_1 = \frac{nW \sin\phi \sin(\theta - a)}{\sin(\phi + \theta)} + W' \sin a \quad (17)$$

$$R_2 = \frac{nW \sin\theta \cos(a + \phi)}{\sin(\phi + \theta)} - W' \cos a$$

$$F_2 = \frac{nW \sin\theta \sin(a - \phi)}{\sin(\phi + \theta)} - W' \sin a;$$

whence we find easily,

$$F_1 + F_2 = nW \frac{\sin(\theta - \phi)}{\sin(\theta + \phi)} \sin a.$$

$$F_1 - F_2 = \frac{2nW \sin\phi \sin\theta \cos a}{\sin(\phi + \theta)} - W \sin a.$$

We must now distinguish three periods, each characterized by a different kind of motion.

(1). When the upper wheel, to which has been communicated a velocity of rotation, is laid upon the supporting wheels, which are initially at rest, the motion at each of the four points of contact is combined slipping and rolling. This motion continues for a certain time, which may be called the first period, terminating when perfect rolling has been established at two of the four points of contact.

(2). The second period commences with the termination of the first, and ends when perfect rolling has been established at the four points of contact.

(3). The third period embraces all the rest of the motion.

It is evident that the equations (13) and (14), depending merely on the immobility of the axes, hold during all these periods.

(1) During the first period, the forces of friction at the several points of contact have their greatest values. We have, therefore,

$$F_1 = R_1 \tan \epsilon', \; F_2 = R_2 \tan \epsilon'$$

where ϵ' is the angle of friction at T_1, T_2. Substituting for R_1 R_2, F_1, F_2 from (17), we find easily

$$nW \sin \phi \sin (\theta - a - \epsilon') + W' \sin (\phi + \theta) \sin (a + \epsilon') = 0,$$

$$nW \sin \theta \sin (a - \phi - \epsilon') - W' \sin (\phi + \theta) \sin (a - \epsilon') = 0.$$

Dividing these equations respectively by

$$\sin \phi \sin \theta \sin(a + \epsilon') \text{ and } \sin \phi \sin \theta \sin (a - \epsilon'),$$

we have

$$\left.\begin{aligned} W' \cot \phi - (W + W') \cot \theta + nW \cot (a + \epsilon') = 0, \\ W' \cot \theta - (W + W') \cot \phi + nW \cot (a - \epsilon') = 0. \end{aligned}\right\} \quad (18)$$

Hence,

$$\left.\begin{aligned} W \cot \phi = W' \cot (a + \epsilon') + (W + W') \cot (a - \epsilon'), \\ W \cot \theta = W' \cot (a - \epsilon') + (W + W') \cot (a + \epsilon'). \end{aligned}\right\} \quad (19)$$

Now, if mk^2 be the moment of inertia of the upper wheel, and ω the angle through which it has revolved, the equation of its rotation will be

$$mk^2 \frac{d^2\omega}{dt^2} = -2b (F_1 + F_2).$$

Substituting for F_1, F_2 from (17), we have

$$mk \frac{d^2\omega}{dt^2} = nWb \sin a \left(\frac{\cot \phi - \cot \theta}{\cot \phi + \cot \theta}\right),$$

or, replacing $\cot \phi$ and $\cot \theta$ by their values (19), we have

$$mk^2 \frac{d^2\omega}{dt^2} = -2Wb \sin a \left\{\frac{\cot (a - \epsilon') - \cot (a + \epsilon')}{\cot (a - \epsilon') + \cot (a + \epsilon')}\right\}$$

$$= -Wb \sin 2\epsilon' \sec a. \quad (20)$$

If then, Ω_0 be the given initial velocity of the upper wheel, we have, by integration,

$$\frac{d\omega}{dt} = \Omega_0 - \frac{Wb \sin 2\epsilon' \sec a}{mk^2} t. \quad (21)$$

Again, substituting for ϕ, θ, in the values of F_1, F_2 (17), we find

$$F_1 = \frac{W \sin \epsilon' \sin (a - \epsilon')}{\sin 2a}, \quad F_2 = \frac{W \sin \epsilon' \sin (a + \epsilon')}{\sin 2a}. \qquad (22)$$

It is plain from these equations that $F_2 > F_1$.

Now, the equations of rotation of the friction wheels being respectively,

$$m'k'^2 \frac{d^2\omega'}{dt^2} = aF_1 - 2\mu r P_1, \text{ and } m'k'^2 \frac{d^2\omega'}{dt^2} = aF_2 - 2\mu r P_2, \quad (23)$$

it is easy to see that, on account of the smallness of r, the velocity of rotation of the wheel $C_2 T_2$ will increase more rapidly than that of $C_1 T_1$. Hence perfect rolling will be established at T_2 sooner than at T_1. Integrating the second of the foregoing equations, we have, putting $r = ia$,

$$m'k'^2 \frac{d\omega'}{dt} = a (F_2 - 2i\mu P_2)\, t.$$

Now, when perfect rolling is established at T_2,

$$a \frac{d\omega'}{dt} = b \frac{d\omega}{dt}.$$

Hence the time at which this takes place is given by the equation,

$$t = \frac{b\Omega_0}{\dfrac{Wb^2 \sin 2\epsilon' \sec a}{mk^2} + \dfrac{a^2}{m'k'^2}(F_2 - 2i\mu P_2)}, \qquad (24)$$

in which the values of $F_2 P_2$ are to be substituted from (22) and (16).

Or, putting $b = i'a$, $m'k'^2 = emk^2$,

$$at = \frac{mk^2 ei'\Omega_0}{Wi'^2 e \sin 2\epsilon' \sec a + F_2 - 2iP^2 \tan \epsilon}. \qquad (25)$$

The velocity of the upper wheel at the moment when perfect rolling is established at T_2 is found by substituting this value of t in (21). This velocity is

$$\frac{d\omega}{dt} = \frac{(F_2 - 2iP_2 \tan \epsilon)\,\Omega_0}{Wi'^2e \sin 2\epsilon' \sec a + F_2 - 2iP_2 \tan \epsilon} = \Omega_1. \quad (26)$$

Hence we have

$$\Omega_0 - \Omega_1 = \frac{Wi'^2e \sin 2\epsilon' \sec a\,\Omega_0}{Wi'^2e \sin 2e' \sec a + F_2 - 2iP^2 \tan \epsilon}.$$

Now, inasmuch as the diameter of the wheel is in general very great as compared with that of the axle, i i' are small fractions. Hence, unless e be large, which is never the case in friction wheels, $\Omega_0 - \Omega_1$ is very small, and may in practice be neglected.

The velocity of the wheel C_1T_1 at the same moment is given by the equation

$$m'k'^2 \frac{d\omega'}{dt} = (F_1 - 2iP_1 \tan \epsilon)\, at$$

$$= \frac{mk^2ei'(F_1 - 2iP_1 \tan \epsilon)\,\Omega_0}{Wi'^2e \sin 2\epsilon' \sec a + F_2 - 2iP_2 \tan \epsilon}.$$

Hence,

$$\frac{d\omega'}{dt} = \frac{i'(F_1 - 2iP_1 \tan \epsilon)\Omega_0.}{Wi'^2e \sin 2\epsilon' \sec a + F_2 - 2iP_2 \tan \epsilon}. \quad (27)$$

Let this velocity be called Ω'_1. It is evidently a small quantity of the same order of magnitude as i'. If quantities of the order i'^2 be neglected, the value of Ω'_1 is

$$\Omega'_1 = i'\Omega_0 \frac{\sin(a - \epsilon')}{\sin(a + \epsilon')}. \quad (28)$$

It is evident from this equation that a cannot be less than ϵ'.

If $a = \epsilon'$, the velocity of the wheel C_1T_1 at the end of the first period is a small quantity of the second order.

We have assumed, in this investigation, that the axis of the upper wheel remains at rest, or, in other words, that the axle does not roll up the circumference of either of the supporting wheels. Now, it is plain from the figure that this rolling up

could only take place at T_2. In order, then, that the axis of the upper wheel may remain at rest, it is necessary and sufficient that the normal pressure R_1 should not vanish. We must have, then, $R_1 > 0$; or, which is evidently the same,

$$F_1 > 0.$$

Substituting the value of F_1 from the first of equations (22), we find,

$$\sin(a - \epsilon') > 0.$$

In order to the immobility of the axis, it is therefore necessary and sufficient that a should exceed the angle of friction at the circumference of the supporting wheel. As the smallest excess is sufficient, we may take ϵ' as the limiting value of a.

(2). During the second period of the motion there is perfect rolling at T_2, and combined rolling and slipping at T_1. The equations applicable to this period are therefore—(1). The equations (16) and (17), which depend solely upon the immobility of the axes. (2). The equation $F_1 = R_1 \tan \epsilon'$, denoting that the force of friction at T_1 has its greatest value. (3). The equation

$$a\frac{d\omega'}{dt} = b\frac{d\omega}{dt},$$

denoting that perfect rolling has been established at T_2. Differentiating this equation, and substituting for the second differential coefficients, we have

$$2iP_2 \tan \epsilon - F_2 = 2ei'^2 (F_1 + F_2),$$

or substituting the values of F_1, F_2, P_2 from (17) and (16),

$$nW \sin\theta \,\{i \sin\epsilon - \sin(a - \phi)\} + W' \sin a \sin(\theta + \phi)$$

$$= 2ei'^2 nW \sin a \sin(\theta - \phi).$$

Dividing by $\sin\theta$, and substituting for $\cot\theta$ from the first of equations (18), we find

$$(n + 1)\, i \sin\epsilon - A \cos\phi + B \sin\phi = 0 \qquad (30)$$

where

$$A = 2(1 - 2ei'^2) \sin a,$$

$$B = (n + 1) \cos a + (n - 1 + 4nei'^2) \sin a \cot (a + \epsilon').$$

This equation determines ϕ. Substituting the value so found in the first equation (18), we obtain the value of θ, and thence those of F_1, F_2, &c.

To find at what instant perfect rolling is established at T_1, we have the equation of rotation of the lower wheel TC_1,

$$m'k'^2 \frac{d^2\omega'}{dt^2} = a\ (F_1 - 2iP_1 \tan \epsilon).$$

Integrating this equation, and observing that at the commencement of the second period, the velocity of rotation of the wheel C_1T_1 is Ω', we have

$$m'k'^2 \left(\frac{d\omega'}{dt} - \Omega'_1\right) = at\ (F_1 - 2iP_1 \tan \epsilon), \qquad (31)$$

t being the time which has elapsed since the commencement of the second period.

Similarly, integrating the equation of rotation of the upper wheel, we have

$$mk^2 \left(\frac{d\omega}{dt} - \Omega_1\right) = -\ 2bt\ (F_1 + F_2) \qquad (32)$$

But when perfect rolling is established at T_1,

$$a\frac{d\omega'}{dt} = b\frac{d\omega}{dt}.$$

Hence,

$$b\Omega_1 - a\Omega'_1 = \left\{\frac{a^2(F_1 - 2iP_1 \tan \epsilon)}{m'k'^2} + \frac{2b^2(F_1 + F_2)}{mk^2}\right\} t \qquad (33)$$

where t is the duration of the second period. The velocity of the upper wheel at the termination of this period is found by eliminating t between (32) and (33). This value, which we shall call Ω_2, is

$$\frac{d\omega}{dt} = \frac{(F_1 - 2iP_1 \tan \epsilon)\,\Omega_1 + 2ei'(F_1 + F_2)\,\Omega'_1}{F_1 - 2iP_1 \tan \epsilon + 2ei'^2(F_1 + F_2)}, \qquad (34)$$

in which F_1, F_2, P_1 are to be replaced by their values as found above. Hence,

$$\Omega_1 - \Omega_2 = \frac{2ei'(F_1 + F_2)(i'\Omega_1 - \Omega'_1)}{F_1 - 2iP_1 \tan \epsilon + 2ei'^2(F_1 + F_2)}.$$

Now, we have seen (p. 193) that Ω'_1 is a small quantity of the order i'. Hence it is evident that $\Omega_1 - \Omega_2$ is (unless e be large) a small quantity of the order i'^2, and may therefore be neglected. We may, therefore, without sensible error, assume that, at the commencement of the third period, the upper wheel preserves the velocity originally communicated to it.

(3). During the third period, perfect rolling being established both at T_1 and T_2, the velocity of rotation must be the same for all the supporting wheels. Hence, evidently,

$$\frac{m'k'^2}{a}\frac{d^2\omega'}{dt^2} = F_1 - 2iP_1 \tan \epsilon = F_2 - 2iP_2 \tan \epsilon. \qquad (35)$$

Substituting for F_1, F_2, P_1, P_2, from (17) and (16), we have

$$2n \sin\theta \sin\phi \cos\alpha - \sin\alpha \sin(\phi + \theta) = ni \sin\epsilon (\sin\phi - \sin\theta). \quad (36)$$

Again, if mk^2 be the moment of inertia of the upper wheel, and ω the angle through which it has revolved, we have

$$mk^2 \frac{d^2\omega}{dt^2} = -2b(F_1 + F_2) = -2nWb\frac{\sin(\theta - \phi)}{\sin(\theta + \phi)}\sin\alpha. \quad (37)$$

Now, since it is supposed that there is no slipping at T_1 or T_2, we have $b\omega = a\omega'$, whence, and from equations (35) and (37),

$$\frac{2b^2nW}{mk^2}\frac{\sin(\theta - \phi)}{\sin(\theta + \phi)}\sin\alpha = \frac{2\mu arP_1 - a^2F_1}{m'k'^2} = \frac{2\mu arP_2 - a^2F_2}{m'k'^2}$$

$$= \frac{2\mu ar(P_1 + P_2) - a^2(F_1 + F_2)}{2m'k'^2}$$

or from (16), (17),

$$= nW\frac{ar(\sin\phi + \sin\theta)\sin\epsilon - a^2\sin(\theta - \phi)\sin\alpha}{2m'k'^2\sin(\theta + \phi)},$$

or

$$\sin\theta + \sin\phi = p\sin(\theta - \phi), \qquad (38)$$

where

$$p = (1 + 4ei'^2)\frac{\sin\alpha}{i\sin\epsilon}.$$

The equations (36) and (38) are sufficient to determine θ, ϕ.

To solve these equations, we may put the second under the form

$$\cos\tfrac{1}{2}(\theta-\phi)\left\{\sin\tfrac{1}{2}(\theta+\phi)-p\sin\tfrac{1}{2}(\theta-\phi)\right\}=0,$$

giving either

$$\theta-\phi=\pi \text{ or } \sin\tfrac{1}{2}(\theta+\phi)=p\sin\tfrac{1}{2}(\theta-\phi).$$

The first of these is inadmissible, inasmuch as we should then have from equations (16),

$$P_1+P_2=0,$$

an impossible equation, as P_1, P_2 are essentially positive quantities. We must, therefore, take the second factor, and put

$$\sin\tfrac{1}{2}(\theta+\phi)=p\sin\tfrac{1}{2}(\theta-\phi). \tag{39}$$

Eliminating, now, $\sin\frac{1}{2}(\theta-\phi)$ from equation (36), which may be put under the form

$$\cos\tfrac{1}{2}(\theta+\phi)\left\{\sin\alpha\sin\tfrac{1}{2}(\theta+\phi)-in\sin\varepsilon\sin\tfrac{1}{2}(\theta-\phi\right\}$$
$$=n\cos\alpha\sin\theta\sin\phi,$$

we have

$$\sin(\theta+\phi)=2q\sin\theta\sin\phi, \tag{40}$$

where

$$q=\frac{pn\cos\alpha}{p\sin\alpha-in\sin\varepsilon}.$$

Assume

$$\sin(\theta-\phi)=v\sin(\theta+\phi). \tag{41}$$

Then, eliminating θ, ϕ between the equations (39), (40), (41), we find

$$v^2=\frac{1}{p^2}+\frac{1}{(p^2-1)\,q^2}. \tag{42}$$

Now, recurring to equation (37), we see that the moment of the force retarding the movement of the upper wheel is

$$2b(F_1+F_2)=2bnvW\sin\alpha.$$

If this wheel rested in fixed bearings, the retarding moment would be, evidently,

$$2bW\tan\varepsilon.$$

Hence, inasmuch as the time required to bring the wheel to rest is inversely proportional to the retarding force, we have,

$$\frac{\text{Time of rotation with friction wheels}}{\text{Time of rotation with fixed bearings}} = \frac{\tan \varepsilon}{nv \sin \alpha}. \quad (43)$$

If, as before, we neglect quantities of the order i^2, i'^2, we have, approximately,

$$p = \frac{\sin \alpha}{i \sin \varepsilon}, \quad q = n \cot \alpha, \quad v^2 = \frac{i^2 \sin^2 \varepsilon}{\sin^2 \alpha}\left(1 + \frac{\tan^2 \alpha}{n^2}\right).$$

Hence, and from (43), we find,

$$\frac{\text{Time of rotation with friction wheels}}{\text{Time of rotation with fixed bearings}} = \frac{1}{i \cos \varepsilon \sqrt{n^2 + \tan^2 \alpha}}. \quad (44)$$

It is evident from this expression that friction wheels are used to most advantage when α is made as small as possible. Now, we have seen (p. 194), that the smallest admissible value of α is ε', or ε, if we suppose the coefficient of friction to be the same everywhere. If, then, in the foregoing expression (44) we make $\alpha = \varepsilon$, we have, finally,

$$\frac{\text{Time of rotation with friction wheels}}{\text{Time of rotation with fixed bearings}} = \frac{1}{i \sqrt{n^2 \cos^2 \varepsilon + \sin^2 \varepsilon}}.$$

Locomotives.

A. *Locomotive with a Single Pair of Driving Wheels.*

7. A locomotive rests upon a number of wheels, of which two only are connected with the engine, these two being firmly attached to the same axle, so as to turn with it. If there be a slight difference between the diameters of these wheels, determine whether either wheel will roll without slipping, and, if so, which.

Let M be the mass of the entire train, F_1, F_2 the effective forces of friction developed at the points of contact of the driving wheels (these forces being reckoned in the direction of the train's motion), and R the entire retarding force, consisting of the resistance of the air, the friction acting on all the other wheels,

and the resolved force of gravity, if the train be ascending an incline. Then the equation of motion of the centre of gravity is

$$M\frac{d^2x}{dt^2} = F_1 + F_2 - R. \qquad (45)$$

Moreover, if mk^2 be the moment of inertia of the driving wheels and axle round their axis of rotation, and θ the angle through which they have revolved during the time t, the equation of rotation is

$$mk^2\frac{d^2\theta}{dt^2} = P - F_1a_1 - F_2a_2, \qquad (46)$$

where P is the moment of the force of steam round the axis of rotation diminished by the moment of the friction of the axle against its supports, and a_1, a_2 are the radii of the driving wheels.

(1). Let it be supposed that $a_1 > a_2$. Then, if the motion of the wheel a_1 be perfect rolling, the motion of the wheel a_2 is mixed slipping and rolling. Moreover, since a_2 is the lesser wheel, it is evident that for this wheel the motion of rotation is *too slow* for perfect rolling. Hence F_2 has its maximum value, and is directed *against* the motion of the train. If, then, W_2 be the pressure which this wheel exerts, and ε the angle of friction, we have

$$F_2 = -W_2\tan\varepsilon, \text{ and } \frac{dx}{dt} = a_1\frac{d\theta}{dt},$$

since the wheel a_1 rolls perfectly. Differentiating this latter equation, and eliminating the differential coefficients from (45) and (46), we have,

$$(ia_1^2 + k^2)F_1 + (ia_1a_2 + k^2)F_2 = k^2R + iPa_1, \qquad (47)$$

putting $M = im$.

Substituting for F_2, and neglecting the small difference between a_1 and a_2, we have, denoting either indifferently by a,

$$F_1 = W_2\tan\varepsilon + \frac{k^2R}{ia^2+k^2} + \frac{iPa}{ia^2+k^2}.$$

Now, we know that F_1 cannot exceed $W_1\tan\varepsilon$, where W_1 is the pressure exerted upon the wheel a_1. We must have, therefore,

$$W_1 = \text{or} > W_2 + \frac{k^2R+iPa}{ia^2+k^2}\cot\varepsilon.$$

But the last term being in general considerable, it is plain that this condition is not fulfilled unless the pressure W_1 be much greater than W_2. If, then, we provide (as we may), that this shall not be the case, the hypothesis of perfect rolling at the circumference of the larger wheel, and, consequently, of a rotation of the smaller wheel too slow for perfect rolling, becomes impossible.

(2). Let $a_1 < a_2$. Then the investigation is the same, with this exception, that inasmuch as the smaller wheel rolls perfectly, the larger wheel must rotate *too fast* for perfect rolling. In this case F_2 acts in the *same* direction as the motion of the train. Hence,

$$F_2 = W_2 \tan \varepsilon.$$

Substituting this value in (47), we have

$$F_1 + W_2 \tan \varepsilon = \frac{k^2R + iPa}{ia^2 + k^2}.$$

Hence, it is evident that the condition requisite for the truth of this hypothesis is

$$W_1 + W_2 = \text{or} > \frac{k^2R + iPa}{ia^2 + k^2} \cot \varepsilon.$$

If this condition be not fulfilled, both wheels will slip, *in consequence of too rapid rotation.* For it is plain that, unless a break be applied to one or both of the driving wheels, the rotation cannot be too slow for *both* wheels. In every case, therefore, the friction developed at the circumferences of both wheels is in the direction of the motion of the train.

The accelerating force by which the train is moved is found by substituting the values of F_1, F_2 in (45). The required value is,

$$\frac{i}{M} \frac{Pa - a^2R}{ia^2 + k^2},$$

so long as there is perfect rolling at the circumference of the smaller wheel. If both wheels be slipping, the accelerating force is

$$\frac{(W_1 + W_2)}{M} \tan \varepsilon - \frac{R}{M}.$$

The greatest steam-tension which can be advantageously used is found by equating these values. We have, then,

$$iPa = (W_1 + W_2)(a^2 + k^2)\tan\epsilon - k^2R, \qquad (48)$$

in which it will be remembered that P denotes the difference between the moment of the steam pressure to turn the driving wheels, and the moment of the friction of the axle in its supports to retard them. Any excess of steam pressure above the value so found is useless.*

B.—*Locomotive with Linked Driving Wheels.*

8. A locomotive rests on $2n$ wheels, which are so attached that they must necessarily revolve with the same angular velocity. If the diameters of these wheels be not mathematically equal, to determine the value of the motion of each, so far as it is determinate.

Before entering upon the special question of the locomotive, we shall consider the following general problem :—

A material system consists of a solid body with a number of wheels attached to it, which are capable of turning round axes fixed in the body, and are parallel to each other. These wheels are so connected that they are compelled to revolve with the same angular velocity, and are acted on by given forces. To determine the equation of rotation.

Let it it be supposed that the connexion between any two of the wheels is effected by means of a coupling bar, attached by two arms of equal length to the axles of the wheels; these arms being firmly fixed to the axles so as to turn with them, and being attached to the coupling bar by a joint round which they can turn freely. For the sake of greater generality, let it be supposed that the weights of these bars are taken into account.

The general dynamical equation for any system is

$$\Sigma m \left\{\left(\frac{d^2x}{dt^2} - X\right)\delta x + \left(\frac{d^2y}{dt} - Y\right)\delta y + \left(\frac{d^2z}{dt^2} - Z\right)\delta z\right\} = 0. \quad (49)$$

Applying this equation to the case of a number of solid bodies connected in any way, we may break up the sum denoted by Σ into a number of integrals, each of which is extended through

* It is, in fact, mischievous.—*Vid.* Note A at the end of the volume.

one of the solid bodies. In the present case, we have three classes of bodies included in Σ; namely, 1. The solid body itself. 2. The attached wheels. 3. The coupling bars. Let, then, M be the mass of the solid body, m that of one of the wheels, and m' that of one of the coupling bars. The general dynamical equation becomes, then,

$$\begin{aligned} 0 = &\int\left\{\left(\frac{d^2x}{dt^2} - X\right)\delta x + \left(\frac{d^2y}{dt^2} - Y\right)\delta y + \left(\frac{d^2z}{dt^2} - Z\right)\delta z\right\} dM \\ &+ \Sigma\int\left\{\left(\frac{d^2x}{dt^2} - X\right)\delta x + \left(\frac{d^2y}{dt^2} - Y\right)\delta y + \left(\frac{d^2z}{dt^2} - Z\right)\delta z\right\} dm \qquad (50) \\ &+ \Sigma\int\left\{\left(\frac{d^2x}{dt^2} - X\right)\delta x + \left(\frac{d^2y}{dt^2} - Y\right)\delta y + \left(\frac{d^2z}{dt^2} - Z\right)\delta z\right\} dm'. \end{aligned}$$

We shall suppose that the motion of the solid body is, at all its points, parallel to a fixed line, which we shall take for the axis of x. We shall suppose, further, that the several axes of rotation are perpendicular to this line, and that the axis of y is parallel to these axes. Let $x_1y_1z_1$ be the co-ordinates of the centre of gravity of the solid body, and $a_1b_1c_1$, $a_2b_2c_2$, &c., the co-ordinates of the centres of gravity of the several wheels. Let, also, $x'y'z'$ be the co-ordinates of any point in the solid body, referred to axes through its centre of gravity, and $\xi\eta\zeta$ the co-ordinates of a point in one of the wheels referred to axes through its centre of gravity. We have, then, for the solid body,

$$x = x_1 + x', \quad y = y_1 + y', \quad z = z_1 + z', \qquad (51)$$

and for the wheel

$$x = a_1 + \xi, \quad y = b_1 + \eta, \quad z = c_1 + \zeta. \qquad (52)$$

We now proceed to consider the values to be assigned to δx, δy, δz for the several parts of the system.

(1). With regard to the solid body itself, as we suppose the motion to be parallel to the axis of x, we may, without limiting the generality of the solution, assume

$$\delta x = \delta x_1, \quad \delta y = 0, \quad \delta z = 0.$$

(2). With respect to the wheels, the rotation being round axes parallel to the axis of y, we may take for the first wheel

$$\delta x = \delta x_1 - \zeta\delta\theta, \quad \delta y = 0, \quad \delta z = \xi\delta\theta,$$

with similar values for the other wheels.

(3). With regard to the coupling bars, it is easy to see that each one of these is at all times perpendicular to the axes of the wheels which it connects, and that each point of the bar describes a circle, whose radius is equal to the common length, b, of the arms, by which the bar is connected with these axes. We may take, therefore, for this part of the system,

$$\delta x = \delta x_1 - b \sin \theta\delta\theta, \quad \delta y = 0, \quad \delta z = b \cos \theta\delta\theta.$$

Substituting these values in the general equation (50), and equating to zero the coefficients of δx_1, $\delta\theta$, we have from the former

$$\left.\begin{aligned}&\int \frac{d^2x}{dt^2} dM + \Sigma \int \frac{d^2x}{dt^2} dm + \Sigma \int \frac{d^2x}{dt^2} dm'. \\ &= \int XdM + \Sigma \int Xdm + \Sigma \int Xdm'.\end{aligned}\right\} \quad (53)$$

If M_0 be the mass of the entire system, and x_0 the x co-ordinate of its centre of gravity, this equation may be written

$$M_0 \frac{d^2x_0}{dt^2} = \int XdM_0.$$

This is the ordinary equation for the motion of the centre of gravity.

The coefficient of $\delta\theta$ being equated to zero gives

$$\left.\begin{aligned}&\Sigma \int \left\{\left(\frac{d^2x}{dt^2} - X\right)\zeta - \left(\frac{d^2z}{dt^2} - Z\right)\xi\right\} dm \\ &+ \Sigma \int \left\{\left(\frac{d^2x}{dt^2} - X\right) b \sin \theta - \left(\frac{d^2z}{dt^2} - Z\right) b \cos \theta\right\} dm' = 0.\end{aligned}\right\} \quad (54)$$

Substituting in the first of these sums the values of x and z from (52), and recollecting that

$$\int \xi dm = 0, \quad \int \zeta dm = 0,$$

we have

$$\int \left(\zeta \frac{d^2x}{dt^2} - \xi \frac{d^2z}{dt^2}\right) dm = \int \left(\zeta \frac{d^2\xi}{dt^2} - \xi \frac{d^2\zeta}{dt^2}\right) dm.$$

Let ρ be the distance of any point in the wheel from its centre. Then

$$\xi = \rho \cos\theta, \quad \zeta = \rho \sin\theta,$$

whence

$$\zeta \frac{d^2\xi}{dt^2} - \xi \frac{d^2\zeta}{dt^2} = -\rho^2 \frac{d^2\theta}{dt^2},$$

and, consequently,

$$\int\left(\zeta \frac{d^2\xi}{dt^2} - \xi \frac{d^2\zeta}{dt^2}\right) dm = -mk^2 \frac{d^2\theta}{dt^2}, \tag{55}$$

where k is the radius of gyration. It is evident, also, that

$$\int(Z\xi - X\zeta)\, dm = L,$$

where L is the moment of the forces acting on the wheel, taken with regard to its axis. The first of the sums denoted by Σ is, therefore,

$$\Sigma L - \frac{d^2\theta}{dt^2} \Sigma mk^2 \tag{56}$$

With respect to the second of these sums, since the motion of each coupling bar is parallel to itself, we have, evidently,

$$\frac{d^2x}{dt^2} = \frac{d^2x_0}{dt^2} + \frac{d^2(b\cos\theta)}{dt^2}, \quad \frac{d^2z}{dt^2} = \frac{d^2(b\sin\theta)}{dt^2},$$

whence,

$$\sin\theta \frac{d^2x}{dt^2} - \cos\theta \frac{d^2z}{dt^2} = \sin\theta \frac{d^2x_0}{dt^2} - b\frac{d^2\theta}{dt^2}.$$

Moreover, as we do not suppose any force, except gravity, to act on the coupling bar, we have, putting a = incl. of axis of x to the horizon,

$$X = -g\sin a, \quad Z = -g\cos a.$$

If these values be substituted in the second of the above sums, it becomes

$$\frac{d^2x_0}{dt^2} \Sigma m'b\sin\theta - \frac{d^2\theta}{dt^2} \Sigma m'b^2 - g\Sigma m'b\cos\theta.$$

Substituting these values in (54), we obtain, finally,

$$\frac{d^2\theta}{dt^2} \Sigma(mk^2 + m'b^2) - \frac{d^2x_0}{dt^2} \Sigma(m'b\sin\theta) - \Sigma L + g\Sigma\{m'b\cos(\theta - a)\} = 0. \tag{57}$$

the values of θ, in the several terms of which Σ is composed, not being necessarily the same, but differing by constant quantities.

If the masses of the coupling bars be so small that they may be neglected, this equation becomes

$$\frac{d^2\theta}{dt^2}\,\Sigma mk^2 = \Sigma L. \qquad (58)$$

being the same as that of the motion of a single wheel whose moment of inertia is the sum of the moments of the linked wheels, and to which the original forces are applied so that the sum of their statical moments with regard to its axis may be equal to the sum of these moments, taken each with regard to the axis of the wheel to which it is actually applied.

The equation

$$\frac{d^2\theta}{dt^2} = \frac{\text{Sum of statical moments}}{\text{Sum of moments of inertia}} \qquad (59)$$

is always true, if we suppose the mass of each coupling bar to be concentrated at the extremity of either arm b, and to be acted on by two accelerating forces; namely, 1. Gravity. 2. The total accelerating force on the system. In the case of a locomotive attached to a train, the second of these forces is, in general, very small as compared with the other, and may in practice be neglected.

9. To apply the equation (59) to the case of a locomotive, let C, C' (Fig. 30), be the centres of two wheels linked by the bar

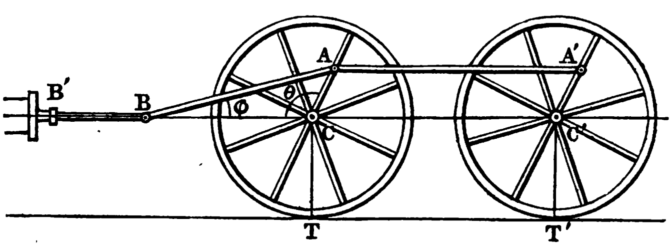

Fig. 30.

AA'. Let the steam pressure, S, be applied in the direction $B'B$. Then, if we put $AB = l$, $AC = A'C' = b$, $ABC = \phi$, $ACB = \theta$, the moment of the steam pressure with regard to C will be

$Sb \cos \phi \sin (\phi + \theta)$, where ϕ and θ are connected by the equation $l \sin \phi = b \sin \theta$. Similarly, for the wheel immediately connected with the engine at the opposite side, the moment will be $S'b \cos \phi' \sin (\phi' + \theta')$, ϕ' and θ' being connected by the same equation as ϕ and θ, and $\theta' - \theta$ being a constant quantity, inasmuch as the wheels revolve at the same rate. The other forces acting on the wheels are the forces of friction $-F_1, -F_2, \ldots -F_{2n}$ acting tangentially at the points T_1, T_2, &c., and the forces of friction $-f_1, -f_2, \ldots -f_{2n}$, acting tangentially at the common surfaces of the axles and their sockets. Let $a_1, a_2, \ldots a_{2n}$ be the radii of the wheels, and $r_1, r_2 \ldots r_{2n}$ the radii of the axles. Then, the sum of the moment of these forces will be

$$-F_1a_1 - F_2a_2 \ldots - F_{2n}a_{2n}, \quad -f_1r_1 - f_2r_2 \ldots - f_{2n}r_{2n},$$

or, neglecting the small differences between the radii, as being here unimportant,

$$-a(F_1 + F_2 +, \&c., + F_{2n}), \quad -r(f_1 + f_2 +, \&c., + f_{2n}).$$

With regard to the terms contained in the sum $\Sigma m'b \cos \theta$, it is plain that the values of θ will be the same for wheels at the same side, differing by a constant from these values for wheels at the opposite side. Hence, putting $2a$ for the constant difference, and M' for the sum of the masses of the bars, we have, assuming the bars at the two sides to be *q. p.* equal

$$\Sigma(m'b \cos \theta) = \frac{1}{2} M'b \{\cos \theta + \cos (\theta + 2a)\} = M'b \cos a \cos (\theta + a).$$

Substituting these terms in the general equation (59), and neglecting the small differences between the wheels, we have the following equation for θ—

$$(2nmk^2 + M'b^2)\frac{d^2\theta}{dt^2} = Sb \cos \phi \sin (\phi + \theta) + S'b \cos \phi' \sin (\phi' + \theta') \tag{60}$$
$$- a(F_1 + F_2 + \&c.) - r(f_1 + f_2 + \&c.) - M'bg \cos a \cos (\theta + a),$$

where ϕ, ϕ', θ' are to be eliminated by the equations

$$l \sin \phi = b \sin \theta, \quad l \sin \phi' = b \sin \theta', \quad \theta' - \theta = 2a.$$

It must be remarked, with regard to this equation, that the term $Sb \cos \phi \sin (\phi + \theta)$ represents the moment of the steam tension only for values of θ contained in the first semicircle. Beyond this it would follow from the trigonometrical formula that the moment should change sign. But the moment of the steam tension never does change sign, inasmuch as the alternating motion of the piston causes the tension itself to change sign at the same time with the perpendicular on its direction. The same is true of the wheel at the opposite side, and the angle θ'.

10. Let, as before, R denote the resultant of the forces resisting the motion of the train, these forces being the resistance of the air and the resolved weight of the train, if aseending an incline. Then the equation of motion of the centre of gravity will be

$$M_0 \frac{d^2 x_0}{dt^2} = F_1 + F_2 + \&c. + F_{2n} - R. \qquad (61)$$

Let it be supposed 1. That there is perfect rolling at the circumference of the wheels 1, 2, . . . p. 2. That the rotation of the wheels $p + 1$, $p + 2$, . . . q, is too rapid for perfect rolling. 3. That the rotation of the wheels $q + 1$, $q + 2$, . . . $2n$ is too slow for perfect rolling.

Then it is plain—1. That the forces $F_1, F_2, \ldots F_p$ will have positive values not exceeding the maximum values μP_1, μP_2, . . . μP_p. 2. That the forces F_{p+1}, $F_{p+2} \ldots F_q$ are also positive, and have their maximum values μP_{p+1}, μP_{p+2}, . . . μP_q. 3. That the forces F_{q+1}, F_{q+2}, . . . F_{2n} are negative, and have their maximum values $-\mu F_{q+1}$, $-\mu F_{q+2}$, . . . $-\mu F_{2n}$. The driving wheels are thus divided into three classes, in the first of which the diameters are equal to each other, while in the second class they are greater, and in the third class less than the diameters of the equal wheels. But inasmuch as these differences are, in general, very small, they will not produce any perceptible effect except that already taken into account in assigning the values and signs to the forces of friction. We may, therefore, in the remaining part of the investigation suppose the diameters of all the wheels to be equal. We may then assume

$$\frac{dx_0}{dt} = a\frac{d\theta}{dt}, \text{ whence } \frac{d^2x_0}{dt^2} = a\frac{d^2\theta}{dt^2}.$$

Hence, eliminating the differential coefficients from the equations (60) and (61), we have

$$\left(a^2 + \frac{2nm}{M_0}k^2 + \frac{M'}{M_0}b^2\right)\Sigma F = ab\,\{S\cos\phi\sin(\phi+\theta) + S'\cos\phi' \tag{62}$$

$$\sin(\theta'+\phi')\} - abM'g\cos a\cos(\theta+a) - ar\,\Sigma f + \frac{R}{M'}(2nmk^2 + M'b^2),$$

putting, for the sake of brevity,

$$\Sigma F = F_1 + F_2 + \&c. + F_{2n},\ \ \Sigma f = f_1 + f_2 + \&c. + f_{2n}.$$

It is plain from this equation that ΣF, the force by which the train is moved forward, is *q. p.* the same, at whatever points perfect rolling takes place, provided that there be perfect rolling at any point. For if we had taken into account the differences between the diameters a_1, a_2, &c., supposing perfect rolling to exist at the circumference of any one of the wheels (a_1 for example), the equation which we should have obtained would have differed from (62) only in having on the left-hand side $a_1(F_1a_1 + F_2a_2 + \&c.)$ instead of $a^2\Sigma F$, and a_1 instead of a on the right-hand side. Hence, it is evident that the value of ΣF thus obtained will differ from that obtained from (62) only by a quantity of the same order as the differences between the diameters.

11. The same conclusion would hold, even if perfect rolling were not established at the circumference of any of the wheels, provided that they were not all slipping in the same direction. For in this case the rotation of some of the wheels would be too rapid, and that of others too slow, for perfect rolling. Hence the equation

$$\frac{dx}{dt} = a\frac{d\theta}{dt}$$

would still be approximately true, and the same conclusions would follow. But this case will never happen. For it is easy to see that in such a case perfect rolling at some one or more of the points of contact is possible, and, therefore, that slipping will not take place at all these points.

If all the points of contact be slipping in the same direction, this direction is necessarily *backwards*, unless a *break* be applied to the driving wheels, which is of course never done. For there is evidently no force sufficient (the friction on the axles and the weight of the coupling bars being small) to prevent the wheels from revolving with sufficient rapidity to cause some of them to roll perfectly. If, therefore, all the wheels slip in the same direction, this slipping must be caused by a velocity of rotation, *too great* in comparison with the velocity of translation. In this case the forces of friction will be all positive, and will have their maximum values.

12. The practical result of the whole discussion is that the utility of linked driving wheels is not sensibly affected by the existence of small differences between their diameters. We may, therefore, safely take advantage of the power which these wheels give us of utilizing the entire pressure of the locomotive for the development of a corresponding force of friction. This power is especially important in the case of a heavy train, for which, in ascending an incline, the force R may be very great, requiring, therefore, to overcome it, a force which a locomotive possessing only a single pair of driving wheels is incapable of exerting.

13. It remains to consider the effect produced by a slight difference of diameter in two wheels, not driving wheels, attached firmly to the same axle.

The equations of Art. 7 are applicable here, omitting the term arising from the steam tension, and supposing R to represent the resultant of all the moving forces which act upon the train or carriage, with the exception of the forces of friction at the circumferences of the wheels, which are denoted by F_1, F_2. As these forces would, if there were perfect rolling at both circumferences, act *against* the motion of the carriage, we shall suppose the forces to be measured in this direction. We have then, adopting the notation of Art. 7,

$$\left.\begin{aligned} M\frac{d^2x}{dt^2} &= R - (F_1 + F_2), \\ mk^2\frac{d^2\theta}{dt^2} &= F_1a_1 + F_2a_2. \end{aligned}\right\} \qquad (63)$$

(1). Let it be supposed that perfect rolling is established at the circumference of the wheel a_1, and let $a_1 > a_2$. Then it is evident that the velocity of rotation of the second wheel, a_2, is *too slow* for perfect rolling, and therefore that the force of friction, F_2, acts *against* the motion of the carriage, or in the direction in which we have supposed these forces to be measured. We must, therefore, take F_2 positively. Eliminating now the differential coefficients as in Art. 7, and neglecting the small difference between a_1 and a_2, we have

$$(k^2 + ia^2)(F_1 + F_2) = Rk^2,$$

whence

$$F_2 = \frac{k^2 R}{k^2 + ia^2} - F_1. \tag{64}$$

Let P_1, P_2 be the pressures on the circumferences of the two wheels respectively. Then, since F_2 has its maximum value,

$$F_2 = P_2 \tan \varepsilon.$$

Hence

$$F_1 = \frac{k^2 R}{k^2 + ia^2} - P_2 \tan \varepsilon$$

Now we know that F_1 cannot lie without the limits $\pm P_1 \tan \varepsilon$. Hence it is necessary to the possibility of the hypothesis of perfect rolling at the circumference of the *larger* wheel that $\dfrac{k^2 R}{k^2 + ia^2}$ should not lie outside the limits $(P_2 + P_1) \tan \varepsilon$, $(P_2 - P_1) \tan \varepsilon$. If $P_1 > P_2$, or, in other words, if the greater pressure correspond to the larger wheel, the latter condition is necessarily fulfilled.

(2). Let $a_1 < a_2$, and let it be supposed, as before, that perfect rolling is established at the circumference of a_1. Then it is evident that the rotation of a_2 is *too fast* for perfect rolling, and therefore that F_2 acts *in the direction* of the motion of the carriage, or in a direction opposite to that in which we have supposed it to be measured. We have then

$$F_2 = -P_2 \tan \varepsilon.$$

Substituting this value in equation (64), which holds equally in this case,

$$F_1 = \frac{k^2 R}{k^2 + ia^2} + P_2 \tan \varepsilon.$$

If, therefore, $\frac{k^2 R}{k^2 + ia^2}$ be greater than $(P_1 - P_2) \tan \varepsilon$, this hypothesis is impossible. This is necessarily so if $P_1 < P_2$, or in other words, if the greater pressure correspond to the larger wheel.

To recapitulate the results of this discussion—

(1). Let it be supposed that the greater pressure corresponds to the larger wheel, and let the moving force R be gradually augmented from zero. Then the larger wheel will roll perfectly until R reaches the limit indicated by the equation

$$\frac{k^2 R}{k^2 + ia^2} = (P_1 + P_2) \tan \varepsilon. \qquad (65)$$

When this limit is passed, both wheels will slip in consequence of *too slow* rotation.

(2). Let it be supposed that the greater pressure corresponds to the smaller wheel, and let R be gradually augmented as before. Then the smaller wheel will roll perfectly until R reaches the limit indicated by the equation

$$\frac{k^2 R}{k^2 + ia^2} = (P_1 - P_2) \tan \varepsilon. \qquad (66)$$

When this limit is passed, the larger wheel will begin to roll perfectly, the smaller wheel slipping until R reaches the limit indicated by (65). If R be still further increased, both wheels will slip as before.

It appears from (64) that the same result holds here as in the case of driving wheels, namely, that the sum of the effective forces of friction developed at the circumferences of two wheels, constituting, in the case of driving wheels, the moving force, and, in the case of supporting wheels, part of the retarding force, is but slightly affected by a small difference between the diameters of the wheels.*

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* The conclusions arrived at in discussing the problem of locomotives are subject to certain modifications, in consequence of the difference between the coefficients of statical and dynamical friction.—*Vid.* Note A.

III.—Questions for Exercise.

1. One end of a beam rests upon a rough plane inclined to the horizon, at an angle greater than the angle of friction; the other end being sustained by a string which, passing over a pulley, supports a weight. If the position of the beam be given—

a. Determine the position of the pulley when the suspended weight is the least possible.

b. Determine the least weight which will support the beam.

c. If the inclination of the plane were equal to or less than the angle of friction, how would these results be modified?

2. Two ladders AB, $A'B'$, resting on a rough horizontal plane AA', are kept in equilibrio by a cord BB' connecting their upper ends. If the friction of the plane be gradually diminished, find which ladder will slip first.

3. A circular cylinder rests upon two planes inclined at the same side of the vertical, having its axes parallel to their intersection, and is supported by a cord coiled round it, passing over a pulley, and sustaining a weight. Determine the conditions of equilibrium.

4. A cylinder is supported on a rough inclined plane by a string coiled round it, and fixed to a point in the plane.

a. If the coefficient of friction be less than unity, determine the point to which the string should be fixed in order that the inclination of the plane may be the greatest possible.

b. Find the greatest inclination of the plane consistent with equilibrium.

5. Determine these positions, if the coefficient of friction be equal to or greater than unity.

6. In the preceding problem, if the string be attached to a weight which hangs freely, find the conditions of equilibrium.

a. If these conditions be not fulfilled, what is requisite in order that the initial motion may be one—

I. Of pure slipping,
II. Of rolling up,
III. Of rolling down.

7. A stick passes over the lower of two rough pegs, and under the other, the pegs being equally rough; find the distance of the centre of gravity of the stick from either peg when equilibrium is on the point of being broken.

8. A rod rests in the limiting position of equilibrium on two equally rough rectangular planes, whose line of intersection is horizontal and at right angles to the rod. If now the position of the planes be changed by causing them (still at right angles to each other) to revolve round their line of intersection, the corresponding positions of the rod will pass through the same point.

9. A cube placed on a smooth inclined plane is kept in equilibrium by a string attached to the middle point of its highest edge, passing over a pulley, and sustaining a weight; show that this is impossible if the inclination of the plane exceed 45°.

10. If the plane be rough and the direction of the string be given, find the least coefficient of friction which will give equilibrium.

11. A material particle is laid on a rough plane inclined to the horizon at the angle of friction; if it be required to drag it down the plane, determine the limits of the direction of the requisite force.

12. A beam resting on two smooth planes is kept in equilibrium by a cord attached to its centre of gravity and to a fixed point; where should the point be situated in order that this may be effected with the weakest possible cord?

13. How is this modified if the planes be rough?

14. Three equal rods AB, BC, CD are connected by smooth hinges at B and C, the extremities AD resting on a rough horizontal plane, and the whole system being situated in a vertical plane. Determine the extreme symmetrical positions of equilibrium.

15. A beam rests with one extremity on a rough horizontal plane, the other extremity being supported against a cylinder of any form, which also lies upon the horizontal plane. Show that the actual amount of friction developed at each point of contact is indeterminate, unless the cylinder be, separately, in a position of equilibrium.

16. A ladder is placed against a rough vertical wall, the coefficients of friction of the wall and the ground being given. Determine the weight of the heaviest man who can ascend to the top of the ladder without causing it to slip, and state under what circumstances this weight is unlimited.

17. A beam is laid horizontally upon two rough planes equally inclined to the horizon, to which their intersection is parallel. If the beam be in an extreme position of equilibrium, show that the angle between the vertical plane passing through it and either of the inclined planes is equal to the complement of the angle of friction, and hence deduce the condition necessary in order that any horizontal position may be a position of equilibrium.

18. It is required to sustain a cubical block on a rough inclined plane, by a cord attached to its upper edge and to a fixed point. Where should the point be placed in order that this may be done with the weakest possible cord?

19. A cubical block is laid on a rough horizontal plane; a string attached to the middle point of one of the upper edges of the cube passes over a pulley and sustains a weight which is gradually increased till equilibrium is broken. Determine the nature of the initial motion.

20. A cubical block, and a cylinder whose diameter is equal to a side of the cube, are laid upon a rough plane, and are attached to each other by a cord coiled round the middle of the cylinder, and fixed to the middle point of one of the edges of the cube which is parallel to the axis of the cylinder. If the plane be then slowly raised (the cube being uppermost) until equilibrium is broken, what will be the nature of the initial motion?

21. A cylinder is laid on a rough horizontal plane, and is in contact with a rough vertical wall, the coefficients of friction being equal; a string, coiled round it at right angles to the axis, passes over a fixed pulley and sustains a weight which is gradually increased till equilibrium is broken. Determine the nature of the initial motion.

22. A heavy bead strung on a rough vertical circle is projected from the lowest point with a velocity just sufficient to carry it round; determine this velocity.

23. A particle resting on a rough inclined plane is attached by a fine string to a fixed point in the plane; if the particle be disturbed from its position of equilibrium, determine the angle through which it must be moved in order that it may just return to its original position.

[N. B.—The original position of equilibrium is supposed to be that which it would be if the plane were smooth.]

24. If a heavy bead, strung on a fixed, rough, circular wire, whose plane is horizontal, receive a given impulse in the direction of the tangent to the circle; determine the arc described before it is reduced to rest.

25. An elastic ball rolls along a rough horizontal plane with a given velocity, and impinges perpendicularly against a smooth vertical wall. Determine the velocity with which it is reflected, and the point of its path at which a perfect rolling motion is re-established.

26. A material particle is attached by a string to a fixed point in a rough inclined plane on which it moves; if the particle start without velocity from the position in which the string is horizontal, determine the greatest coefficient of friction which would allow it to reach the lowest point.

27. A cylinder is laid on a rough inclined plane to a point in which, situated above the point of contact, it is attached by a string coiled round it at right angles to the axis, and leaving it at a point which is not one of contact with the plane; if the cylinder be allowed to descend, find the accelerating force on its centre of gravity in any given position.

28. A right cone is laid on an inclined plane, so rough as to prevent slipping, and is slightly disturbed from its position of equilibrium. Determine the motion.

29 A weight is laid on an imperfectly rough inclined plane; a thin cord, attached to the weight, passes under and round a cylinder which is laid on the same plane above the weight; the

coefficients of friction being different for the two bodies. Excluding lateral movements, the simple movements of which each of these bodies is capable are—for the first, slipping up or down; and for the second—1. pure rolling up or down; 2. rolling and slipping up or down; 3. pure slipping up or down; 4. rotation without motion of the centre of gravity. Examine the various combinations of these movements, and determine which of them are possible.

30. A sphere is moving on a horizontal plate (so rough as to prevent slipping) which has a uniform motion of rotation round a vertical axis. Determine the motion.

31. A sphere descends a rough inclined plane of given height and length, under the influence of gravity. Show that there is necessarily a value of the coefficient of friction for which the *vis viva* gained in the descent is a *minimum*.

32. Two spheres are projected with different velocities and in different directions along a rough horizontal plane. Show that the motion has three periods, during the first of which the path of the centre of gravity is an arc of a parabola, during the second an arc of a different parabola, and during the third a right line.

33. A cylinder of any form is laid upon a rough horizontal plane, and allowed to oscillate. Prove that the initial motion will be pure rolling if

$$\tan \alpha \tan \epsilon > \text{or} = \frac{p^2}{k^2 + p^2}$$

where ϵ is the angle of friction, α the angle which the plane passing through the centre of gravity of the cylinder and the line of contact makes with the horizon, p the vertical height of the centre of gravity, and k the radius of gyration of the cylinder round an axis parallel to the sides, and passing through the centre of gravity.

34. A carriage resting on four equal wheels rolls down an

inclined plane; determine the time in which it will reach the bottom.

35. A circular cylinder is laid upon two others which are placed upon a rough inclined plane, the axes of all three being horizontal. Show that equilibrium is impossible, and determine the nature of the initial movement.

36. It is required to draw a heavy body up a rough inclined plane by the application of a constant force parallel to the plane. Determine the intensity, so that the total expenditure of the force may be the least possible.

NOTE A.

STATICAL AND DYNAMICAL FRICTION.

In all questions which concern the transition of a particle from a state of rest to a state of motion, the results obtained must be modified by the consideration that the coefficient of friction is changed in this passage. Thus, for example, a force applied to a material particle which rests upon a rough surface may be sufficient to keep it in motion, and even to communicate to it a finite velocity when it has once begun to move, and yet may be insufficient to set it in motion. The first object will be attained if the ratio of the tangential to the normal component of the acting force (supposed to be finite) exceed, by any finite quantity, the coefficient of dynamical friction. For the attainment of the second object, it is necessary that the same ratio should exceed the coefficient of statical friction, which differs sensibly from the former.

It is plain, therefore, that the problem of initial motion differs sensibly according as we suppose the velocities initially communicated to a particle or system to be absolutely zero, or to be only indefinitely small. This difference is due to two causes: the first (which has been already considered in pp. 104-9) being the fact that the geometrical forces change sensibly in the transition of the system from a state of rest to a state of motion; the second being the change, also sensible, which, in this transition, affects the coefficient of friction. Both causes tend to produce the same result—namely, the possible development of a *finite* force by the communication of indefinitely small velocities to a system which was previously in equilibrium.

A result of some importance to Practical Mechanics follows from these considerations. If we wish to utilise fully the force of friction as a resisting power, we must be careful not to augment the ratio of the tangential to the normal component of the force acting at the point of contact sufficiently far to produce relative motion. For, if relative motion be produced, the available force of friction is at once sensibly diminished.

The same result will often follow an *unsteady* application of force. A slight blow, or a sudden muscular contraction, may, by augmenting the tangential component, produce a relative velocity, or *slipping*, at

the point of contact. Now, if there were no difference between statical and dynamical friction, this velocity, which is in general small, would be unimportant, as it would be speedily destroyed by the forces which had previously produced equilibrium. But, how small soever the velocity be, it is sufficient to convert statical into dynamical friction, and therefore to diminish sensibly the force by which equilibrium is preserved. It is, therefore, quite possible that the tangential component of the acting force, which, in the state of rest, was less than the maximum force of friction, may, in the state of motion, exceed it, and thus augment the small velocity originally communicated. This seems to be the explanation of the advantage which belongs to a steady step in walking over slippery ground.

The same conclusion is applicable to the transition of a solid body from a motion of pure rolling to a motion of combined rolling and slipping. So long as the motion is pure rolling, the point of contact is at each instant at rest, and therefore the friction developed is statical friction. The limit of the force so developed is therefore the maximum value of statical friction, and we may utilize this or any smaller amount. But when the point of contact begins to slip, the friction becomes dynamical, and its maximum value is sensibly reduced. If, therefore, we desire that the amount of friction developed at the point of contact should be as great as possible, we must be careful so to arrange the moving force that the motion of the body may be one of pure rolling.

Now, this is precisely the case with a locomotive. The force by which the train is drawn forward is necessarily equal to the sum of the forces of friction developed at the points of contact of the driving wheels. In order, therefore, that this sum may be as great as possible, we must endeavour to provide that the motion of these wheels shall be a motion of pure rolling. If, then, the motion of rotation impressed by the engine be too rapid for pure rolling, the force by which the train is drawn forward will be augmented by reducing the steam pressure until perfect rolling is established.

Practical engineers are perfectly aware of this principle, which they express by saying that, when the wheel begins to slip, its *bite* is diminished.

NOTE B, p. 170.

The following difficulty, connected with Ex. 2, may suggest itself to the student:—

If the system be in equilibrium, this equilibrium will not be disturbed by fixing the lower point, A. If this point be fixed, the only possible movement of the system will be a rotation round A. Such a rotation, which would lower the centre of gravity, does not at first sight appear to be resisted by any force whatever. For at the very commencement of the rotation the rod CD would separate from the vertical board at the point E. As this separation would be effected without any motion of the one body *along* the surface of the other, no friction would be developed at the point E; and there would be nothing to prevent the rotation from taking place.

This reasoning would be conclusive, if we assumed the perfect rigidity of the system. If there be absolutely *no* elasticity in the rods, or in the joint at B, the initial movement of CD would necessarily separate it from the vertical board, and the force of friction could not, as is evident from its nature, resist this separation. But if there be *any* elasticity in the system, no matter how small the amount may be, the initial movement of CD will *not* separate it from the vertical board. When the system is placed as in the figure, the angle ABE is slightly increased by the strains at A and E. As AB revolves round A, the elasticity of the joint will tend to restore ABE to its original value, and thus cause BC to remain in contact with the board for a time, which is not absolutely zero. There will be, thus, a movement of sliding of BC along the board at E, which movement will necessarily develope and be resisted by the force of friction at this point. This result is obviously independent of the amount of elasticity possessed by the rods, provided that it be not absolutely zero.

THE END.